Plastic Analysis
and Design of Structures

Part 3

Linear Programming in Limit Analysis Using Kinematic Approach :

Reference External Force : $\bar{\mathbf{f}}$

– Kinematic Equation : $\dot{\mathbf{e}} = \mathbf{B}\dot{\mathbf{d}}$

– Kinematically Admissible Condition : $\bar{\mathbf{f}}^T \dot{\mathbf{d}} > 0$

– Power Equality : $\mu_k \bar{\mathbf{f}}^T \dot{\mathbf{d}} = \mathbf{s}_0^T \left|\dot{\mathbf{e}}\right|$

– Upper Bound Theorm : $Minimize\ \mu_k = \dfrac{\mathbf{s}_0^T \left|\dot{\mathbf{e}}\right|}{\bar{\mathbf{f}}^T \dot{\mathbf{d}}}$

Suppose : $\bar{\mathbf{f}}^T \dot{\mathbf{d}} = 1$

Linear Programming in Limit Analysis Using Kinematic Approach :

$$Minimize\ \mu_k = \mathbf{s}_0^T |\dot{\mathbf{e}}| \rightarrow Objective\ Function$$

$$\left.\begin{aligned} &\dot{\mathbf{e}} - \mathbf{B}\dot{\mathbf{d}} = \mathbf{0} \\ &\bar{\mathbf{f}}^T \dot{\mathbf{d}} = 1 \end{aligned}\right\} \rightarrow Constraints$$

$$\dot{\mathbf{e}}^+ = \frac{|\dot{\mathbf{e}}| + \dot{\mathbf{e}}}{2} \rightarrow \dot{\mathbf{e}}^+ \geq \mathbf{0}$$

$$\dot{\mathbf{e}}^- = \frac{|\dot{\mathbf{e}}| - \dot{\mathbf{e}}}{2} \rightarrow \dot{\mathbf{e}}^- \geq \mathbf{0}$$

$$\left.\begin{aligned} &\dot{\mathbf{e}}^+ \geq \mathbf{0} \\ &\dot{\mathbf{e}}^- \geq \mathbf{0} \end{aligned}\right\} \rightarrow \left\{\begin{aligned} &\dot{\mathbf{e}} = \dot{\mathbf{e}}^+ - \dot{\mathbf{e}}^- \\ &|\dot{\mathbf{e}}| = \dot{\mathbf{e}}^+ + \dot{\mathbf{e}}^- \end{aligned}\right.$$

Linear Programming in Limit Analysis Using Kinematic Approach :

$$\left\{\begin{array}{l} Minimize\ \mu_k = \mathbf{s}_0^T \dot{\mathbf{e}}^+ + \mathbf{s}_0^T \dot{\mathbf{e}}^- \\ \dot{\mathbf{e}}^+ - \dot{\mathbf{e}}^- - \mathbf{B}\dot{\mathbf{d}} = \mathbf{0} \\ \bar{\mathbf{f}}^T \dot{\mathbf{d}} = 1 \\ \dot{\mathbf{e}}^+ \geq \mathbf{0} \\ \dot{\mathbf{e}}^- \geq \mathbf{0} \end{array}\right.$$

$$\dot{\mathbf{d}}^+ = \frac{|\dot{\mathbf{d}}| + \dot{\mathbf{d}}}{2} \rightarrow \dot{\mathbf{d}}^+ \geq \mathbf{0}$$

$$\dot{\mathbf{d}}^- = \frac{|\dot{\mathbf{d}}| - \dot{\mathbf{d}}}{2} \rightarrow \dot{\mathbf{d}}^- \geq \mathbf{0}$$

$$\left.\begin{array}{l} \dot{\mathbf{d}}^+ \geq \mathbf{0} \\ \dot{\mathbf{d}}^- \geq \mathbf{0} \end{array}\right\} \rightarrow \dot{\mathbf{d}} = \dot{\mathbf{d}}^+ - \dot{\mathbf{d}}^-$$

Linear Programming in Limit Analysis Using Kinematic Approach :

$$\begin{cases} \textit{Minimize } \mu_k = \mathbf{s}_0^T \dot{\mathbf{e}}^+ + \mathbf{s}_0^T \dot{\mathbf{e}}^- \\ \dot{\mathbf{e}}^+ - \dot{\mathbf{e}}^- - \mathbf{B}\dot{\mathbf{d}}^+ + \mathbf{B}\dot{\mathbf{d}}^- = \mathbf{0} \\ \overline{\mathbf{f}}^T \dot{\mathbf{d}}^+ - \overline{\mathbf{f}}^T \dot{\mathbf{d}}^- = 1 \\ \dot{\mathbf{e}}^+ \geq \mathbf{0} \\ \dot{\mathbf{e}}^- \geq \mathbf{0} \\ \dot{\mathbf{d}}^+ \geq \mathbf{0} \\ \dot{\mathbf{d}}^- \geq \mathbf{0} \end{cases}$$

Linear Programming in Limit Analysis Using Kinematic Approach :

$$\mathbf{I}\dot{\mathbf{e}} = \mathbf{B}\dot{\mathbf{d}} \quad \longrightarrow \quad \mathbf{I}_{m\times m}\dot{\mathbf{e}}_m = \mathbf{B}_{m\times n}\dot{\mathbf{d}}_n \;;\; n = m - s$$

$$\Big[\;\mathbf{I}\;\Big]_{m\times m}\Big\{\dot{\mathbf{e}}\Big\}_m = \Big[\;\mathbf{B}\;\Big]_{m\times n}\Big\{\dot{\mathbf{d}}\Big\}_n$$

$$\begin{array}{c} \left[\begin{array}{c} \mathbf{P}^T \\ \hline \mathbf{S}^T \end{array}\right] \end{array}\begin{array}{l} {}_{n\times m} \\ {}_{(m-n)\times m} \end{array} \Big\{\dot{\mathbf{e}}\Big\} = \left[\begin{array}{c} \mathbf{I} \\ \hline \mathbf{0} \end{array}\right]\begin{array}{l} {}_{n\times n} \\ {}_{(m-n)\times n} \end{array} \Big\{\dot{\mathbf{d}}\Big\} \qquad \begin{cases} \mathbf{P}^T\dot{\mathbf{e}} = \dot{\mathbf{d}} \\ \mathbf{S}^T\dot{\mathbf{e}} = \mathbf{0} \end{cases}$$

Linear Programming in Limit Analysis Using Kinematic Approach :

$$\begin{cases} \text{Minimize } \mu_k = \mathbf{s}_0^T \dot{\mathbf{e}}^+ + \mathbf{s}_0^T \dot{\mathbf{e}}^- \\ \dot{\mathbf{e}}^+ - \dot{\mathbf{e}}^- - \mathbf{B}\dot{\mathbf{d}} = \mathbf{0} \\ \overline{\mathbf{f}}^T \dot{\mathbf{d}} = 1 \\ \dot{\mathbf{e}}^+ \geq \mathbf{0} \\ \dot{\mathbf{e}}^- \geq \mathbf{0} \end{cases}$$

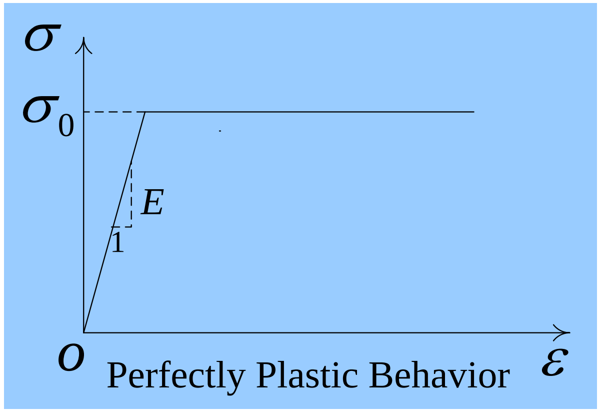
$$\begin{cases} \mathbf{P}^T \dot{\mathbf{e}} = \dot{\mathbf{d}} \\ \mathbf{S}^T \dot{\mathbf{e}} = \mathbf{0} \end{cases}$$

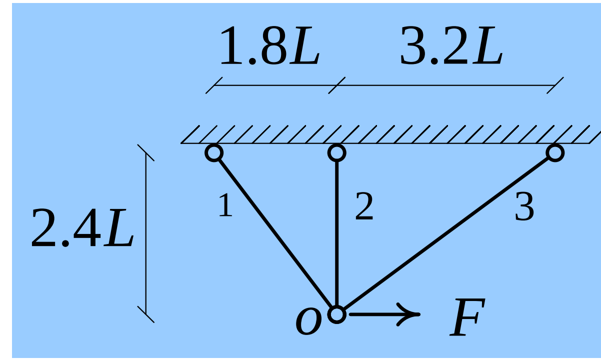
$$\begin{cases} \text{Minimize } \mu_k = \mathbf{s}_0^T \dot{\mathbf{e}}^+ + \mathbf{s}_0^T \dot{\mathbf{e}}^- \\ \mathbf{S}^T \dot{\mathbf{e}}^+ - \mathbf{S}^T \dot{\mathbf{e}}^- = \mathbf{0} \\ \overline{\mathbf{f}}^T \mathbf{P}^T \dot{\mathbf{e}}^+ - \overline{\mathbf{f}}^T \mathbf{P}^T \dot{\mathbf{e}}^- = 1 \\ \dot{\mathbf{e}}^+ \geq \mathbf{0} \\ \dot{\mathbf{e}}^- \geq \mathbf{0} \end{cases}$$

$$\mathbf{S}^T \dot{\mathbf{e}}^+ - \mathbf{S}^T \dot{\mathbf{e}}^- = \mathbf{0} \longrightarrow \textit{Equation of Compatibility}$$

Example:

$$\begin{cases} A_1 = 0.90A \\ A_2 = 0.96A \\ A_3 = A \end{cases}$$





$$\begin{cases} \sigma_{y1} = 0.8\sigma_0 \\ \sigma_{y2} = 0.06\sigma_0 \\ \sigma_{y3} = \sigma_0 \end{cases}$$

$$\sigma_0 A = S_0$$

$$\mathbf{s}_0 = \begin{Bmatrix} 0.72 \\ 0.0576 \\ 1 \end{Bmatrix} S_0$$

$$\begin{cases} \vec{U}_{xo} = u \\ \downarrow U_{yo} = v \end{cases}$$

$$\mathbf{d} = \begin{Bmatrix} u \\ v \end{Bmatrix}$$

$$\bar{\mathbf{f}} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

$$\mathbf{e} = \begin{Bmatrix} e_1 \\ e_2 \\ e_3 \end{Bmatrix}$$

$$Kinematic\ Equations: \begin{cases} \dot{e}_1 = 0.6\dot{u} + 0.8\dot{v} \\ \dot{e}_2 = \dot{v} \\ \dot{e}_3 = -0.8\dot{u} + 0.6\dot{v} \end{cases}$$

Example, continue…:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{Bmatrix} = \begin{bmatrix} 0.6 & 0.8 \\ 0 & 1 \\ -0.8 & 0.6 \end{bmatrix} \begin{Bmatrix} \dot{u} \\ \dot{v} \end{Bmatrix}$$

$$\begin{bmatrix} \frac{1}{0.6} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{Bmatrix} = \begin{bmatrix} 1 & \frac{0.8}{0.6} \\ 0 & 1 \\ -0.8 & 0.6 \end{bmatrix} \begin{Bmatrix} \dot{u} \\ \dot{v} \end{Bmatrix}$$

$$\begin{bmatrix} \frac{1}{0.6} & -\frac{0.8}{0.6} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -0.8 & 0.6 \end{bmatrix} \begin{Bmatrix} \dot{u} \\ \dot{v} \end{Bmatrix}$$

$$\begin{bmatrix} \frac{1}{0.6} & -\frac{0.8}{0.6} & 0 \\ 0 & 1 & 0 \\ \frac{0.8}{0.6} & -\frac{1}{0.6} & 1 \end{bmatrix} \begin{Bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{u} \\ \dot{v} \end{Bmatrix}$$

Example, continue…:

$$\begin{bmatrix} \frac{1}{0.6} & -\frac{0.8}{0.6} & 0 \\ 0 & 1 & 0 \\ \frac{0.8}{0.6} & -\frac{1}{0.6} & 1 \end{bmatrix} \begin{Bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{u} \\ \dot{v} \end{Bmatrix}$$

$$\begin{bmatrix} \frac{1}{0.6} & -\frac{0.8}{0.6} & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{Bmatrix} = \begin{Bmatrix} \dot{u} \\ \dot{v} \end{Bmatrix}$$

$$\begin{bmatrix} \frac{0.8}{0.6} & -\frac{1}{0.6} & 1 \end{bmatrix} \begin{Bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{Bmatrix} = 0$$

$$\bar{\mathbf{f}}^T \dot{\mathbf{d}} = 1$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{Bmatrix} \dot{u} \\ \dot{v} \end{Bmatrix} = 1$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{0.6} & -\frac{0.8}{0.6} & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{Bmatrix} = 1$$

$$\begin{bmatrix} \frac{1}{0.6} & -\frac{0.8}{0.6} & 0 \end{bmatrix} \begin{Bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{Bmatrix} = 1$$

$$Minimize\ \mu_k = S_0 \begin{bmatrix} 0.72 & 0.0576 & 1 \end{bmatrix} \begin{Bmatrix} |\dot{\mathbf{e}}_1| \\ |\dot{\mathbf{e}}_2| \\ |\dot{\mathbf{e}}_3| \end{Bmatrix}$$

Example, continue…:

$$Minimize\ \mu_k = S_0 \begin{bmatrix} 0.72 & 0.0576 & 1 \end{bmatrix} \begin{Bmatrix} \left|\dot{\mathbf{e}}_1\right| \\ \left|\dot{\mathbf{e}}_2\right| \\ \left|\dot{\mathbf{e}}_3\right| \end{Bmatrix}$$

$$\begin{bmatrix} \dfrac{0.8}{0.6} & -\dfrac{1}{0.6} & 1 \end{bmatrix} \begin{Bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{Bmatrix} = 0$$

$$\begin{bmatrix} \dfrac{1}{0.6} & -\dfrac{0.8}{0.6} & 0 \end{bmatrix} \begin{Bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{Bmatrix} = 1$$

$\dfrac{-\mu_k}{S_0}$	$\dot{e}_1^+$	$\dot{e}_2^+$	$\dot{e}_3^+$	$\dot{e}_1^-$	$\dot{e}_2^-$	$\dot{e}_3^-$	RHS
0	0.8	-1	0.6	-0.8	1	-0.6	0
0	1	-0.8	0	-1	0.8	0	0.6
1	0.72	0.0576	1	0.72	0.0576	1	0

Example, continue…:

	$\frac{-\mu_k}{S_0}$	$\dot{e}_1^+$	$\dot{e}_2^+$	$\dot{e}_3^+$	$\dot{e}_1^-$	$\dot{e}_2^-$	$\dot{e}_3^-$	RHS
$\dot{e}_2^+$	0	0	1	-1.6667	0	-1	1.6667	1.3333
$\dot{e}_1^+$	0	1	0	-1.3333	-1	0	1.3333	1.6667
	1	0	0	2.0560	1.44	0.1152	-0.056	-1.2768

$$\begin{cases} \dot{e}_1^+ = 1.6667 \\ \dot{e}_2^+ = 1.3333 \\ \dot{e}_3^+ = 0 \\ \dot{e}_1^- = 0 \\ \dot{e}_2^- = 0 \\ \dot{e}_3^- = 0 \end{cases}$$

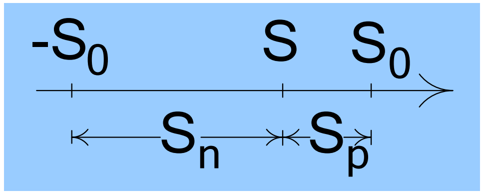
$$\frac{-\mu_k}{S_0} = -1.2768 \rightarrow \mu_k = 1.2768 S_0$$

Example, continue…:

	$\frac{-\mu_k}{S_0}$	$\dot{e}_1^+$	$\dot{e}_2^+$	$\dot{e}_3^+$	$\dot{e}_1^-$	$\dot{e}_2^-$	$\dot{e}_3^-$	RHS
$\dot{e}_3^-$	0	0	0.6	-1	0	-0.6	1	0.8
$\dot{e}_1^+$	0	1	-0.8	0	-1	0.8	0	0.6
	1	0	0.0336	2	1.44	0.0816	0	-1.232

$$\begin{cases} \dot{e}_1^+ = 0.6 \\ \dot{e}_2^+ = 0 \\ \dot{e}_3^+ = 0 \\ \dot{e}_1^- = 0 \\ \dot{e}_2^- = 0 \\ \dot{e}_3^- = 0.8 \end{cases}$$

$$\frac{-\mu_k}{S_0} = -1.232 \quad \longrightarrow \quad \mu_k = 1.232 S_0$$



$$S_p = S_0 - S \quad \longrightarrow \quad \begin{cases} 0 = 0.72S_0 - S_1 \rightarrow S_1 = 0.72S_0 \\ 0.0336S_0 = 0.0576S_0 - S_2 \rightarrow S_2 = 0.024S_0 \\ 2S_0 = S_0 - S_3 \rightarrow S_3 = -S_0 \end{cases}$$

$$S_n = S - (-S_0) \quad \longrightarrow \quad \begin{cases} 1.44S_0 = 0.72S_0 + S_1 \rightarrow S_1 = 0.72S_0 \\ 0.0816S_0 = 0.0576S_0 + S_2 \rightarrow S_2 = 0.024S_0 \\ 0 = S_0 + S_3 \rightarrow S_3 = -S_0 \end{cases}$$

Linear Programming in Limit Analysis Using Kinematic Approach

Application in Frames:

$$\dot{\mathbf{e}} = \mathbf{B}\dot{\mathbf{d}}$$

$$\begin{Bmatrix} \dot{\mathbf{e}}_a \\ \dot{\mathbf{e}}_b \end{Bmatrix} = \begin{Bmatrix} \mathbf{B}_a \\ \mathbf{B}_b \end{Bmatrix} \dot{\mathbf{d}}$$

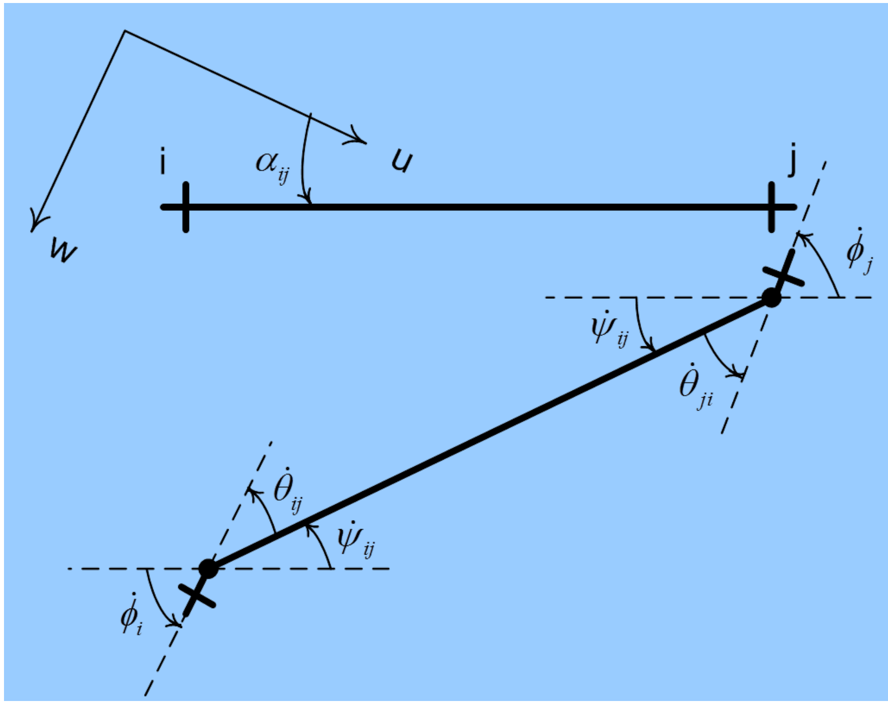
$$\begin{Bmatrix} \mathbf{0} \\ \dot{\mathbf{e}}_b \end{Bmatrix} = \begin{Bmatrix} \mathbf{B}_a \\ \mathbf{B}_b \end{Bmatrix} \dot{\mathbf{d}}$$

$$\begin{Bmatrix} \mathbf{0} \\ \mathbf{I} \end{Bmatrix} \dot{\mathbf{e}}_b = \begin{Bmatrix} \mathbf{B}_a \\ \mathbf{B}_b \end{Bmatrix} \dot{\mathbf{d}}$$

$$\begin{array}{cc} \left[\begin{array}{c} \mathbf{P}^T \\ \hline \mathbf{S}^T \end{array}\right] & \begin{array}{l} n\times m \\ (m-n)\times m \end{array} \end{array} \begin{Bmatrix} \dot{\mathbf{e}}_b \end{Bmatrix} = \begin{array}{cc} \left[\begin{array}{c} \mathbf{I} \\ \hline \mathbf{0} \end{array}\right] & \begin{array}{l} n\times n \\ (m-n)\times n \end{array} \end{array} \begin{Bmatrix} \dot{\mathbf{d}} \end{Bmatrix}$$

$$\left\{ \begin{array}{l} \mathbf{P}^T \dot{\mathbf{e}}_b = \dot{\mathbf{d}} \\ \mathbf{S}^T \dot{\mathbf{e}}_b = \mathbf{0} \end{array} \right.$$

Definition of Frames Plastic Deformations:



-Plastic Extension:

$$\dot{e}_{ij} = (\dot{u}_j - \dot{u}_i)\cos\alpha_{ij} + (\dot{w}_j - \dot{w}_i)\sin\alpha_{ij} = 0$$

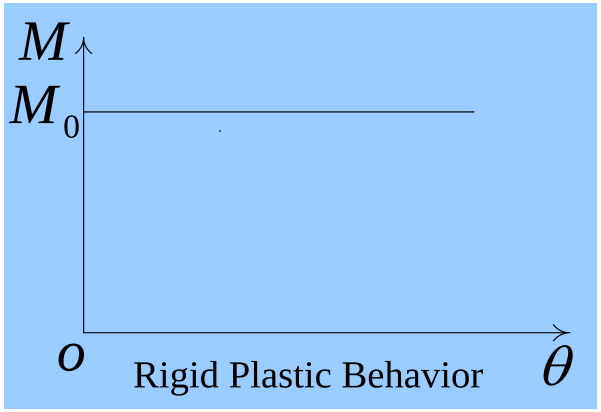
$$\dot{\psi}_{ij} = \frac{\dot{u}_j - \dot{u}_i}{l_{ij}}\sin\alpha_{ij} - \frac{\dot{w}_j - \dot{w}_i}{l_{ij}}\cos\alpha_{ij}$$

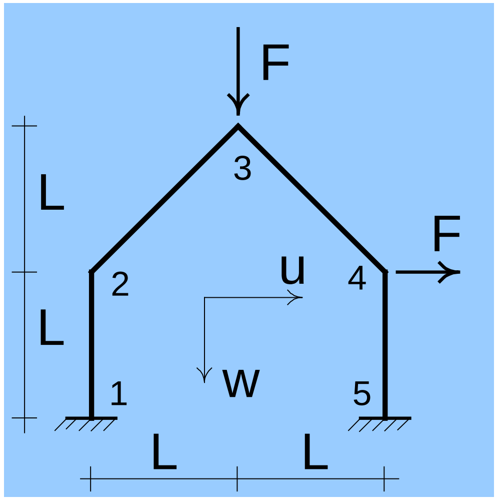
-Plastic Rotations:

$$\dot{\theta}_{ij} = \dot{\phi}_i - \dot{\psi}_{ij} = \dot{\phi}_i - \left(\frac{\dot{u}_j - \dot{u}_i}{l_{ij}}\sin\alpha_{ij} - \frac{\dot{w}_j - \dot{w}_i}{l_{ij}}\cos\alpha_{ij}\right)$$

$$\dot{\theta}_{ji} = \dot{\phi}_j - \dot{\psi}_{ij} = \dot{\phi}_j - \left(\frac{\dot{u}_j - \dot{u}_i}{l_{ij}}\sin\alpha_{ij} - \frac{\dot{w}_j - \dot{w}_i}{l_{ij}}\cos\alpha_{ij}\right)$$

Example:





$$\mathbf{d} = \begin{Bmatrix} u_2 \\ w_2 \\ \phi_2 \\ u_3 \\ w_3 \\ \phi_3 \\ u_4 \\ w_4 \\ \phi_4 \end{Bmatrix}$$

$$\mathbf{e} = \begin{Bmatrix} e_{12} \\ e_{23} \\ e_{34} \\ e_{45} \\ \theta_{12} \\ \theta_{21} \\ \theta_{23} \\ \theta_{32} \\ \theta_{34} \\ \theta_{43} \\ \theta_{45} \\ \theta_{54} \end{Bmatrix}$$

$$\bar{\mathbf{f}} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix}$$

Example, continue…:

$$
\underbrace{\begin{Bmatrix} \cancel{\dot{e}_{12}} \\ \cancel{\dot{e}_{23}} \\ \cancel{\dot{e}_{34}} \\ \cancel{\dot{e}_{45}} \\ \dot{\theta}_{12} \\ \dot{\theta}_{21} \\ \dot{\theta}_{23} \\ \dot{\theta}_{32} \\ \dot{\theta}_{34} \\ \dot{\theta}_{43} \\ \dot{\theta}_{45} \\ \dot{\theta}_{54} \end{Bmatrix}}_{\dot{\mathbf{e}}}
=
\underbrace{\begin{bmatrix}
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.7071 & 0.7071 & 0 & 0.7071 & -0.7071 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.7071 & -0.7071 & 0 & 0.7071 & 0.7071 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
1/L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/L & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1/L & -1/L & 1 & 1/L & 1/L & 0 & 0 & 0 & 0 \\
-1/L & -1/L & 0 & 1/L & 1/L & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/L & -1/L & 1 & -1/L & 1/L & 0 \\
0 & 0 & 0 & 1/L & -1/L & 0 & -1/L & 1/L & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/L & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/L & 0 & 0
\end{bmatrix}}_{\mathbf{B}}
\underbrace{\begin{Bmatrix} \dot{u}_2 \\ \dot{w}_2 \\ \dot{\phi}_2 \\ \dot{u}_3 \\ \dot{w}_3 \\ \dot{\phi}_3 \\ \dot{u}_4 \\ \dot{w}_4 \\ \dot{\phi}_4 \end{Bmatrix}}_{\dot{\mathbf{d}}}
$$

Example, continue…:

$$\underbrace{\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \dot{\theta}_{12} \\ \dot{\theta}_{21} \\ \dot{\theta}_{23} \\ \dot{\theta}_{32} \\ \dot{\theta}_{34} \\ \dot{\theta}_{43} \\ \dot{\theta}_{45} \\ \dot{\theta}_{54} \end{Bmatrix}}_{\dot{\mathbf{e}}} = \underbrace{\begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.7071 & 0.7071 & 0 & 0.7071 & -0.7071 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.7071 & -0.7071 & 0 & 0.7071 & 0.7071 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1/L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/L & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1/L & -1/L & 1 & 1/L & 1/L & 0 & 0 & 0 & 0 \\ -1/L & -1/L & 0 & 1/L & 1/L & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/L & -1/L & 1 & -1/L & 1/L & 0 \\ 0 & 0 & 0 & 1/L & -1/L & 0 & -1/L & 1/L & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/L & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/L & 0 & 0 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{Bmatrix} \dot{u}_2 \\ \dot{w}_2 \\ \dot{\phi}_2 \\ \dot{u}_3 \\ \dot{w}_3 \\ \dot{\phi}_3 \\ \dot{u}_4 \\ \dot{w}_4 \\ \dot{\phi}_4 \end{Bmatrix}}_{\dot{\mathbf{d}}}$$

Example, continue…:

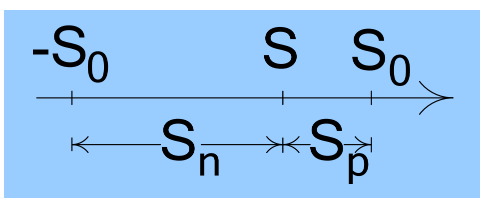
$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\underbrace{\begin{Bmatrix}
\dot{\theta}_{12} \\ \dot{\theta}_{21} \\ \dot{\theta}_{23} \\ \dot{\theta}_{32} \\ \dot{\theta}_{34} \\ \dot{\theta}_{43} \\ \dot{\theta}_{45} \\ \dot{\theta}_{54}
\end{Bmatrix}}_{\dot{\mathbf{e}}_b}
=
\underbrace{\begin{bmatrix}
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.7071 & 0.7071 & 0 & 0.7071 & -0.7071 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.7071 & -0.7071 & 0 & 0.7071 & 0.7071 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
1/L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/L & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1/L & -1/L & 1 & 1/L & 1/L & 0 & 0 & 0 & 0 \\
-1/L & -1/L & 0 & 1/L & 1/L & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/L & -1/L & 1 & -1/L & 1/L & 0 \\
0 & 0 & 0 & 1/L & -1/L & 0 & -1/L & 1/L & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/L & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/L & 0 & 0
\end{bmatrix}}_{\mathbf{B}}
\underbrace{\begin{Bmatrix}
\dot{u}_2 \\ \dot{w}_2 \\ \dot{\phi}_2 \\ \dot{u}_3 \\ \dot{w}_3 \\ \dot{\phi}_3 \\ \dot{u}_4 \\ \dot{w}_4 \\ \dot{\phi}_4
\end{Bmatrix}}_{\dot{\mathbf{d}}}
$$

Linear Programming in Limit Analysis Using Static Approach :

$- \mathit{Equilibrium\ \ Equation:}$ $$\mu_s \bar{\mathbf{f}} = \mathbf{B}^T \mathbf{s}$$

$- \mathit{Plastically\ Admissible\ Condition:}$ $$-\mathbf{s}_0 \leq \mathbf{s} \leq \mathbf{s}_0$$

$- \mathit{Lower\ Bound\ Theorm:}$ $$\mathit{Maximize}: \mu_s$$



$$\left.\begin{array}{l} \mathbf{s}_p = \mathbf{s}_0 - \mathbf{s} \geq 0 \\ \\ \mathbf{s}_n = \mathbf{s}_0 + \mathbf{s} \geq 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \mathbf{s} = \dfrac{\mathbf{s}_n - \mathbf{s}_p}{2} \\ \\ \mathbf{s}_0 = \dfrac{\mathbf{s}_n + \mathbf{s}_p}{2} \end{array}\right.$$

Linear Programming in Limit Analysis Using Static Approach :

$$Minimize: -\mu_s$$

$$-\mathbf{B}^T\mathbf{s}_p + \mathbf{B}^T\mathbf{s}_n - 2\mu_s\bar{\mathbf{f}} = \mathbf{0}$$

$$\mathbf{s}_p + \mathbf{s}_n = 2\mathbf{s}_0$$

$$\mathbf{s}_p \geq \mathbf{0}$$

$$\mathbf{s}_n \geq \mathbf{0}$$

$$\mu_s \geq 0$$

Linear Programming in Limit Analysis Using Static Approach :

$-\,Equilibrium \quad Equation:$

$$\mu_s \bar{\mathbf{f}} = \mathbf{B}^T \mathbf{s}$$

$-\,Plastically\ Admissible\ Condition:$

$$-\mathbf{s}_0 \le \mathbf{s} \le \mathbf{s}_0$$

$-\,Lower\ Bound\ Theorm:$

$$Maximize: \mu_s$$

$$\left.\begin{array}{l} \mathbf{s}^+ = \dfrac{|\mathbf{s}| - \mathbf{s}}{2} \ge 0 \\ \mathbf{s}^- = \dfrac{|\mathbf{s}| - \mathbf{s}}{2} \ge 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \mathbf{s} = \mathbf{s}^+ - \mathbf{s}^- \\ |\mathbf{s}| = \mathbf{s}^+ + \mathbf{s}^- \le \mathbf{s}_0 \end{array}\right.$$

$$|\mathbf{s}| = \mathbf{s}^+ + \mathbf{s}^- \le \mathbf{s}_0 \Rightarrow \left\{\begin{array}{l} \mathbf{s}^+ + \mathbf{s}^- + \Delta\mathbf{s} = \mathbf{s}_0 \\ \Delta\mathbf{s} = \mathbf{s}_0 - |\mathbf{s}| \ge 0 \end{array}\right.$$

Linear Programming in Limit Analysis Using Static Approach :

$$Minimize: -\mu_s$$

$$\mathbf{B}^T\mathbf{s}^+ - \mathbf{B}^T\mathbf{s}^- - \mu_s\bar{\mathbf{f}} = \mathbf{0}$$

$$\mathbf{s}^+ + \mathbf{s}^- + \Delta\mathbf{s} = \mathbf{s}_0$$

$$\mathbf{s}^+ \geq \mathbf{0}$$

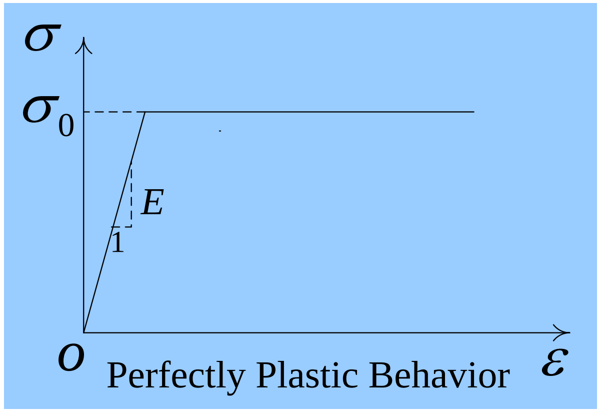
$$\mathbf{s}^- \geq \mathbf{0}$$

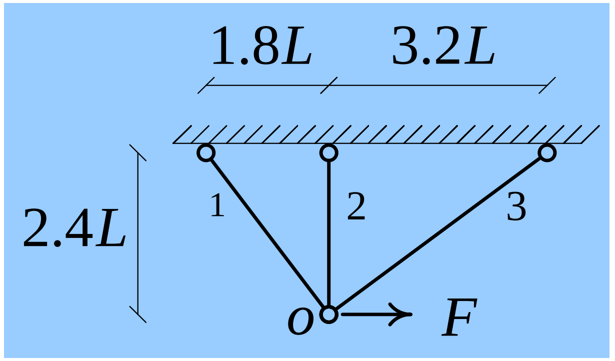
$$\Delta\mathbf{s} \geq \mathbf{0}$$

$$\mu_s \geq 0$$

Example:

$$\begin{cases} A_1 = 0.90A \\ A_2 = 0.96A \\ A_3 = A \end{cases}$$





$$\begin{cases} \sigma_{y1} = 0.8\sigma_0 \\ \sigma_{y2} = 0.06\sigma_0 \\ \sigma_{y3} = \sigma_0 \end{cases}$$

$$\sigma_0 A = S_0$$

$$\mathbf{s}_0 = \begin{Bmatrix} 0.72 \\ 0.0576 \\ 1 \end{Bmatrix} S_0 = \begin{Bmatrix} S_1^+ + S_1^- + \Delta S_1 \\ S_2^+ + S_2^- + \Delta S_2 \\ S_3^+ + S_3^- + \Delta S_3 \end{Bmatrix}$$

$$\bar{\mathbf{f}} = \begin{Bmatrix} S_0 \\ 0 \end{Bmatrix} \qquad \mathbf{f} = \mu_s \begin{Bmatrix} S_0 \\ 0 \end{Bmatrix}$$

$$Equilibrium\ Equations: \begin{cases} \sum F_x = F \\ \sum F_y = 0 \end{cases} \rightarrow \begin{cases} 0.6S_1 - 0.8S_3 = \mu_s S_0 \\ 0.8S_1 + S_2 + 0.6S_3 = 0 \end{cases}$$

Example, continue…:

$$Equilibrium\ Equations: \begin{cases} \sum F_x = F \\ \sum F_y = 0 \end{cases} \rightarrow \begin{cases} 0.6S_1 - 0.8S_3 = \mu_s S_0 \\ 0.8S_1 + S_2 + 0.6S_3 = 0 \end{cases}$$

$$\begin{bmatrix} 0.6 & 0 & -0.8 \\ 0.8 & 1 & 0.6 \end{bmatrix} \begin{Bmatrix} S_1^+ - S_1^- \\ S_2^+ - S_2^- \\ S_3^+ - S_3^- \end{Bmatrix} - \mu_s \begin{Bmatrix} S_0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

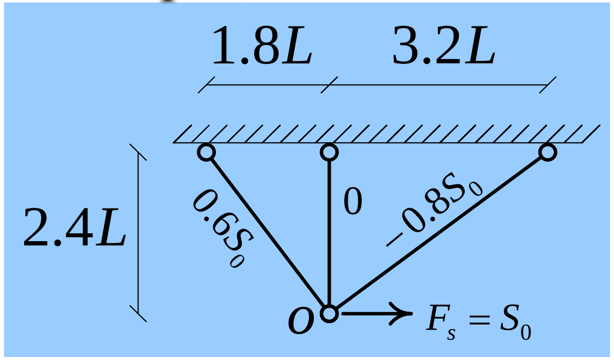
Example, continue…:

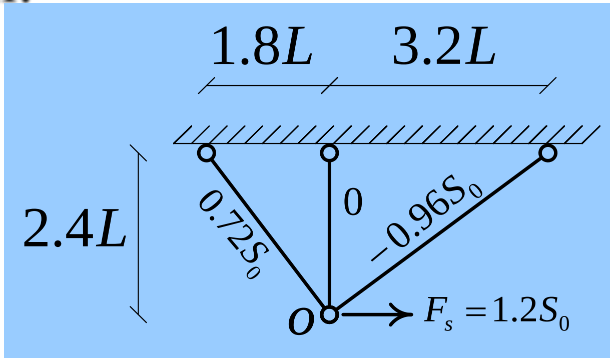
$$\mathbf{s}_0 = \begin{Bmatrix} 0.72 \\ 0.0576 \\ 1 \end{Bmatrix} S_0 = \begin{Bmatrix} S_1^+ + S_1^- + \Delta S_1 \\ S_2^+ + S_2^- + \Delta S_2 \\ S_3^+ + S_3^- + \Delta S_3 \end{Bmatrix}$$

$$\begin{bmatrix} 0.6 & 0 & -0.8 \\ 0.8 & 1 & 0.6 \end{bmatrix} \begin{Bmatrix} S_1^+ - S_1^- \\ S_2^+ - S_2^- \\ S_3^+ - S_3^- \end{Bmatrix} - \mu_s \begin{Bmatrix} S_0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

μ_s	$\frac{S_1^+}{S_0}$	$\frac{S_2^+}{S_0}$	$\frac{S_3^+}{S_0}$	$\frac{S_1^-}{S_0}$	$\frac{S_2^-}{S_0}$	$\frac{S_3^-}{S_0}$	$\frac{\Delta S_1}{S_0}$	$\frac{\Delta S_2}{S_0}$	$\frac{\Delta S_3}{S_0}$	*RHS*
0	1	0	0	1	0	0	1	0	0	0.72
0	0	1	0	0	1	0	0	1	0	0.0576
0	0	0	1	0	0	1	0	0	1	1
-1	0.6	0	-0.8	-0.6	0	0.8	0	0	0	0
0	0.8	1	0.6	-0.8	-1	-0.6	0	0	0	0
-1	0	0	0	0	0	0	0	0	0	0

Example, continue…: -Feasible Solution:





$$
\begin{cases}
S_1^+ = 0.6S_0 \\
S_2^+ = 0 \\
S_3^+ = 0 \\
S_1^- = 0 \\
S_2^- = 0 \\
S_3^- = 0.8S_0 \\
\Delta S_1 = 0.12S_0 \\
\Delta S_2 = 0.0576S_0 \\
\Delta S_3 = 0.2S_0 \\
\mu_s = 1.0
\end{cases}
$$

-Number of Constraints: 5

-Number of Basic Variables: 5

-Number of Variables: 10

$$
\begin{cases}
S_1^+ = 0.72S_0 \\
S_2^+ = 0 \\
S_3^+ = 0 \\
S_1^- = 0 \\
S_2^- = 0 \\
S_3^- = 0.96S_0 \\
\Delta S_1 = 0 \\
\Delta S_2 = 0.0576S_0 \\
\Delta S_3 = 0.04S_0 \\
\mu_s = 1.2
\end{cases}
$$

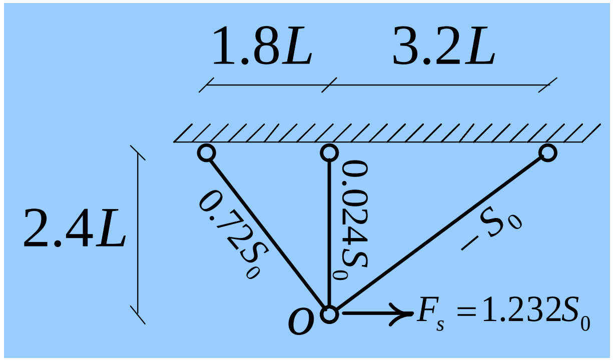
	μ_s	$\frac{S_1^+}{S_0}$	$\frac{S_2^+}{S_0}$	$\frac{S_3^+}{S_0}$	$\frac{S_1^-}{S_0}$	$\frac{S_2^-}{S_0}$	$\frac{S_3^-}{S_0}$	$\frac{\Delta S_1}{S_0}$	$\frac{\Delta S_2}{S_0}$	$\frac{\Delta S_3}{S_0}$	*RHS*
S_1^+	0	1	0	0	1	0	0	1	0	0	0.72
ΔS_2	0	0	1	0	0	1	0	0	1	0	0.0576
ΔS_3	0	0	1.667	2	-2.667	-1.667	0	-1.333	0	1	0.04
μ_s	1	0	-1.333	0	3.333	1.333	0	1.667	0	0	1.2
S_3^-	0	0	-1.667	-1	2.667	1.667	1	1.333	0	0	0.96
	0	0	-1.333	0	3.333	1.333	0	1.667	0	0	1.2

	μ_s	$\frac{S_1^+}{S_0}$	$\frac{S_2^+}{S_0}$	$\frac{S_3^+}{S_0}$	$\frac{S_1^-}{S_0}$	$\frac{S_2^-}{S_0}$	$\frac{S_3^-}{S_0}$	$\frac{\Delta S_1}{S_0}$	$\frac{\Delta S_2}{S_0}$	$\frac{\Delta S_3}{S_0}$	*RHS*
S_1^+	0	1	0	0	1	0	0	1	0	0	0.72
ΔS_2	0	0	0	-1.2	1.6	2	0	0.8	1	-0.6	0.0336
S_2^+	0	0	1	1.2	-1.6	-1	0	-0.8	0	0.6	0.024
μ_s	1	0	0	1.6	1.2	0	0	0.6	0	0.8	1.232
S_3^-	0	0	0	1	0	0	1	0	0	1	1
	0	0	0	1.6	1.2	0	0	0.6	0	0.8	1.232

Example, continue…:

-Optimum Solution:

$$
\begin{cases}
S_1^+ = 0.72S_0 \\
S_2^+ = 0.024S_0 \\
S_3^+ = 0 \\
S_1^- = 0 \\
S_2^- = 0 \\
S_3^- = S_0 \\
\Delta S_1 = 0 \\
\Delta S_2 = 0.0336S_0 \\
\Delta S_3 = 0 \\
\mu_s = 1.232
\end{cases}
$$



Linear Programming in Shakedown Analysis Using Static Approach :

$- Equilibrium \quad Equation:$

$$\mathbf{B}^T \mathbf{s}_r = \mathbf{0}$$

$- Shakedown\,Condition:$

$$\begin{cases} \mathbf{s}_r + \mu_s \mathbf{s}_{e\min} \le \mathbf{s}_0 \\ \mathbf{s}_r + \mu_s \mathbf{s}_{e\max} \ge -\mathbf{s}_0 \end{cases}$$

$- Lower\ Bound\ Theorm:$

$$Maximize: \mu_s$$

$$\left.\begin{array}{l} \mathbf{s}_r^+ = \dfrac{\left|\mathbf{s}_r\right| - \mathbf{s}_r}{2} \ge 0 \\[2em] \mathbf{s}_r^- = \dfrac{\left|\mathbf{s}_r\right| - \mathbf{s}_r}{2} \ge 0 \end{array}\right\} \Rightarrow \mathbf{s}_r = \mathbf{s}_r^+ - \mathbf{s}_r^-$$

$$\begin{cases} \mathbf{s}_r^+ - \mathbf{s}_r^- + \mu_s \mathbf{s}_{e\min} + \Delta\mathbf{s}_{\min} = \mathbf{s}_0 \\ \mathbf{s}_r^+ - \mathbf{s}_r^- + \mu_s \mathbf{s}_{e\max} - \Delta\mathbf{s}_{\max} = -\mathbf{s}_0 \end{cases}$$

Linear Programming in Shakedown Analysis Using Static Approach :

$$Minimize: -\mu_s$$

$$\mathbf{B}^T\mathbf{s}_r^+ - \mathbf{B}^T\mathbf{s}_r^- = \mathbf{0}$$

$$\mathbf{s}_r^+ - \mathbf{s}_r^- + \mu_s\mathbf{s}_{e\min} + \Delta\mathbf{s}_{\min} = \mathbf{s}_0$$

$$\mathbf{s}_r^+ - \mathbf{s}_r^- + \mu_s\mathbf{s}_{e\max} - \Delta\mathbf{s}_{\max} = -\mathbf{s}_0$$

$$\mathbf{s}_r^+ \geq \mathbf{0}$$

$$\mathbf{s}_r^- \geq \mathbf{0}$$

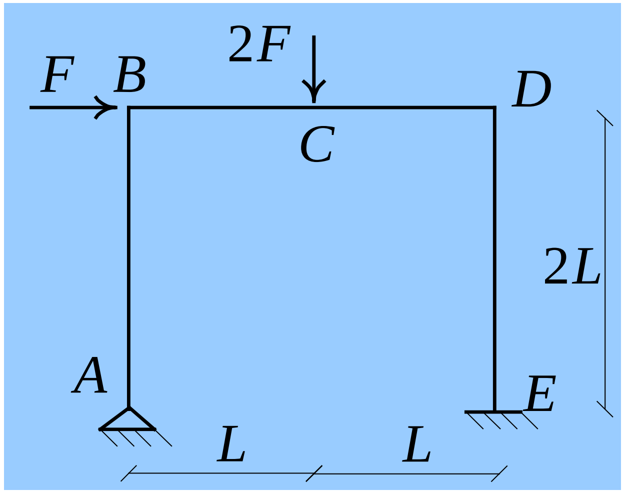
$$\Delta\mathbf{s}_{\min} \geq \mathbf{0}$$

$$\Delta\mathbf{s}_{\max} \geq \mathbf{0}$$

$$\mu_s \geq 0$$

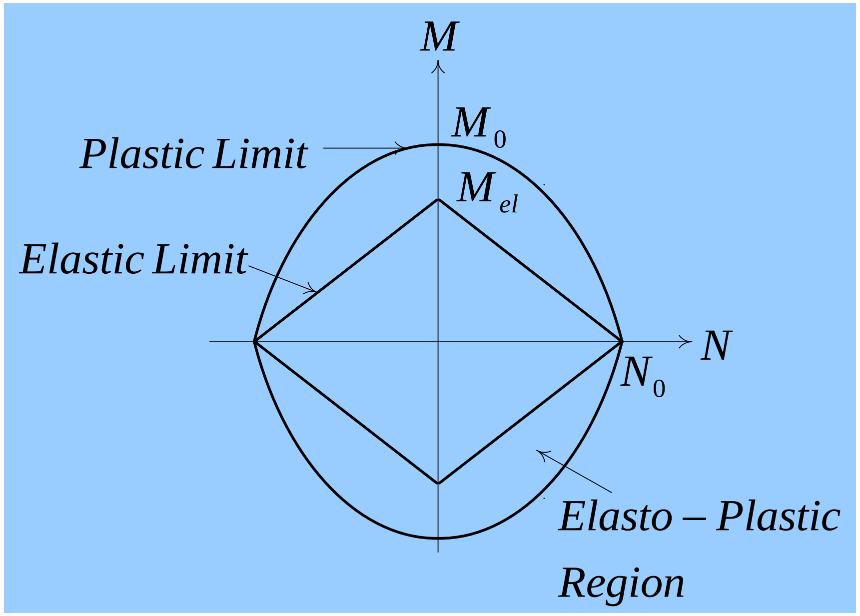
Limit Analysis in Frames, Considering the Effect of Normal Forces:

-Simple Estimate of Collapse Load, example:



Elastic Limit : $$\frac{|M|}{M_{el}}+\frac{|N|}{N_0}=1$$

Plastic Limit : $$\frac{|M|}{M_0}+\left(\frac{N}{N_0}\right)^2=1$$



Example, continue…:

Plastic Limit :

$$\frac{|M|}{M_0} + \left(\frac{N}{N_0}\right)^2 = 1$$

Suppose :

$$h_0 = \frac{M_0}{N_0}$$

$$\lambda_0 = \frac{L}{h_0}$$

Plastic Limit :

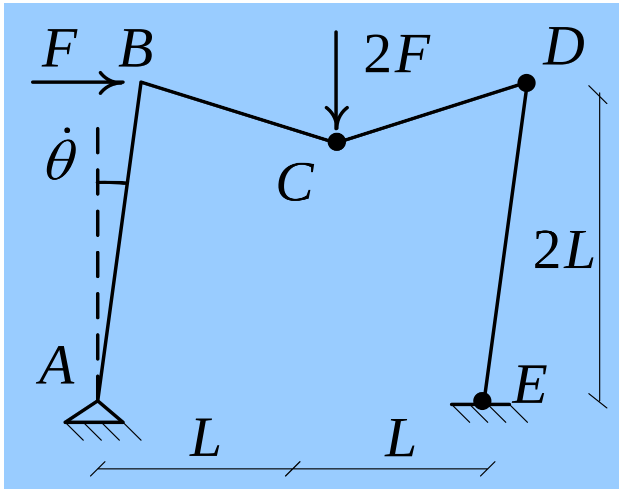
$$\frac{|M|}{M_0} + \left(\frac{h_0 N}{M_0}\right)^2 = 1$$

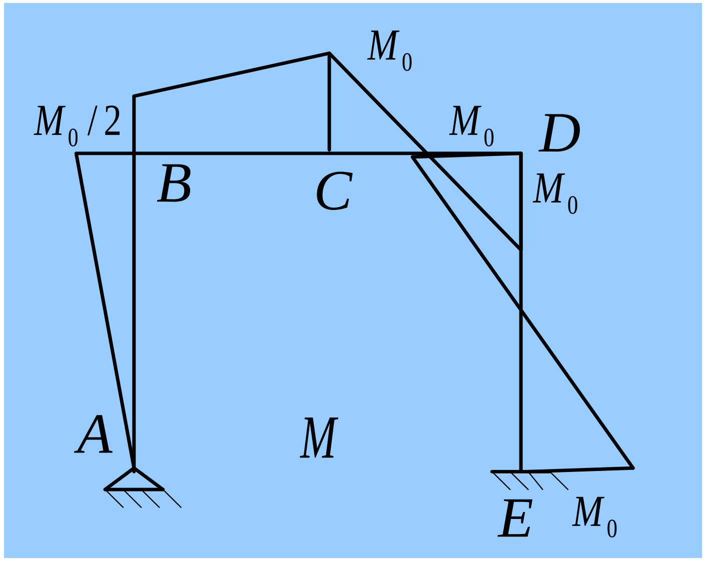
$$\frac{|M|}{M_0} + \left(\frac{LN}{\lambda_0 M_0}\right)^2 = 1$$

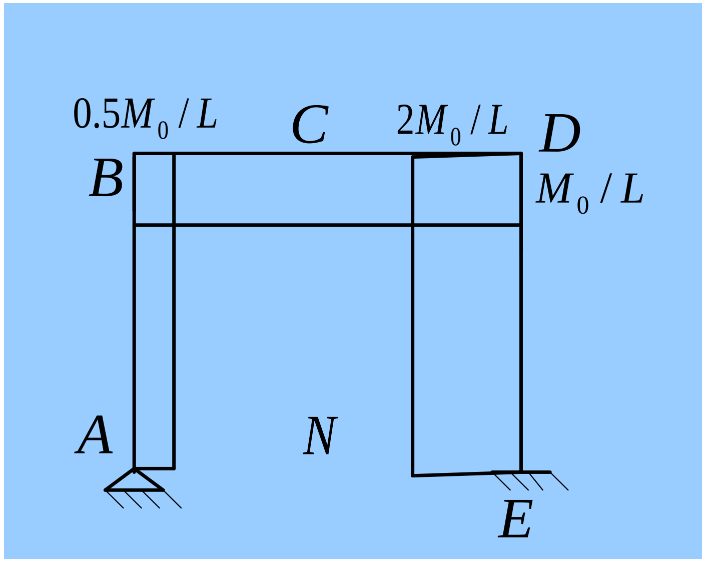
Example, continue…:

-The Collapse Mechanism Without Accounting the Effect of Normal Forces:

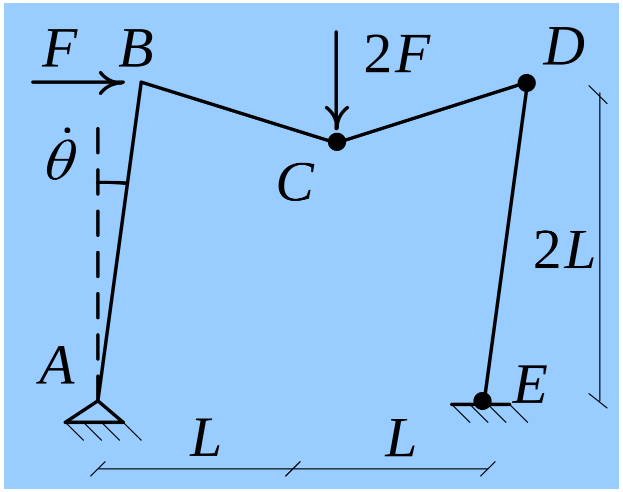
$$F = \frac{1.25M_0}{L}$$





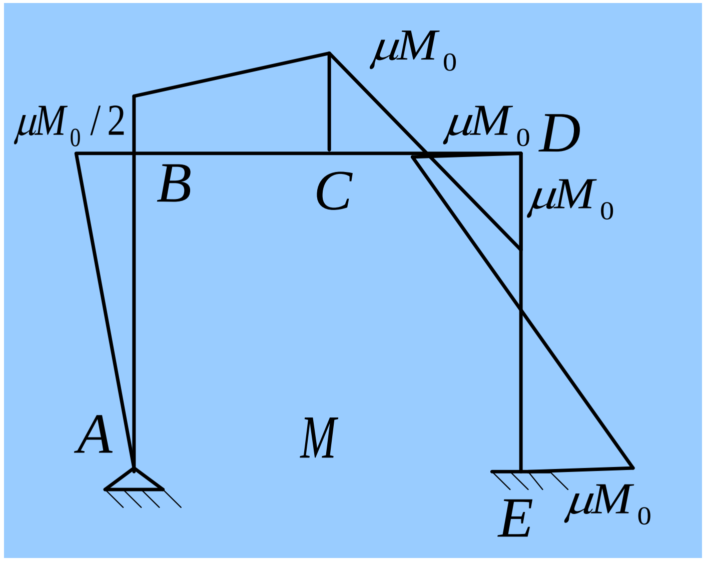


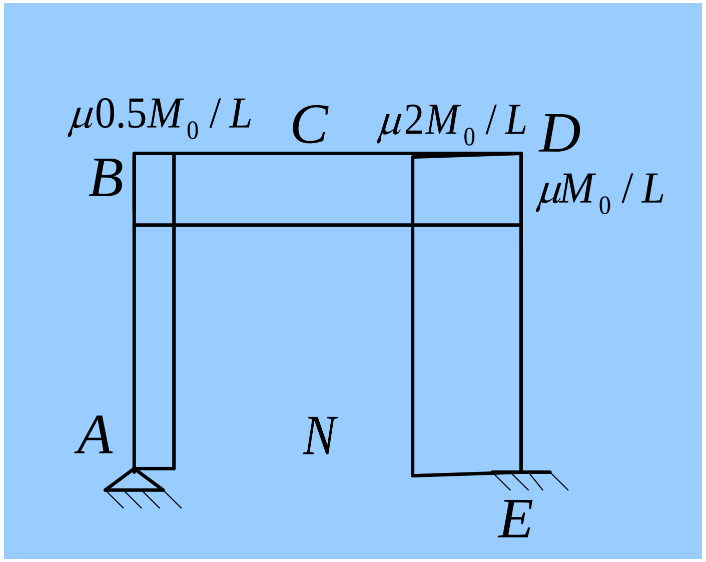
Example, continue…:



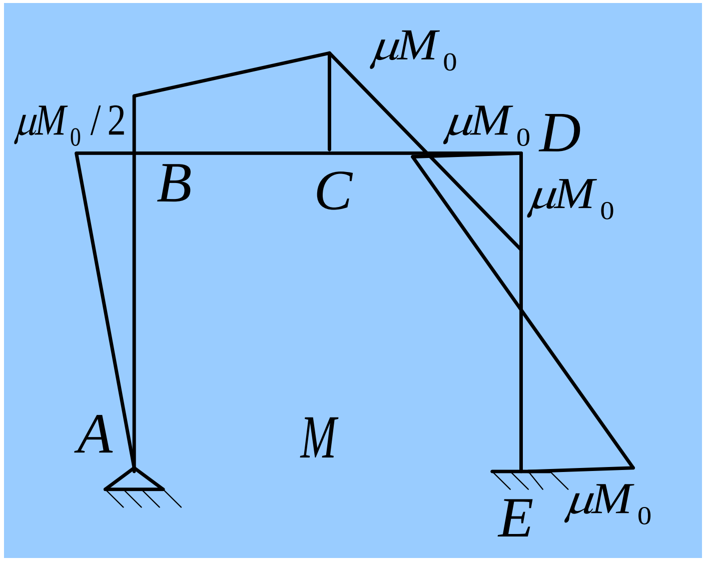
-The Collapse Mechanism Considering The Effect of Normal Forces:

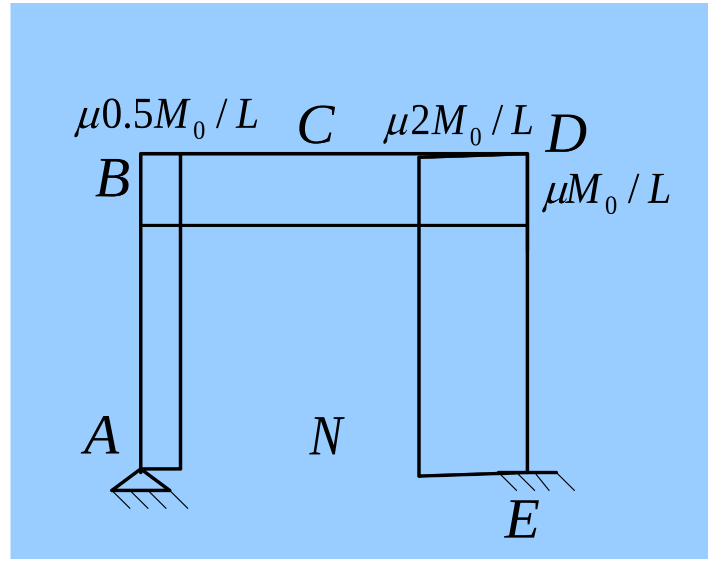
$$F = \mu \frac{1.25 M_0}{L}$$





Example, continue…:





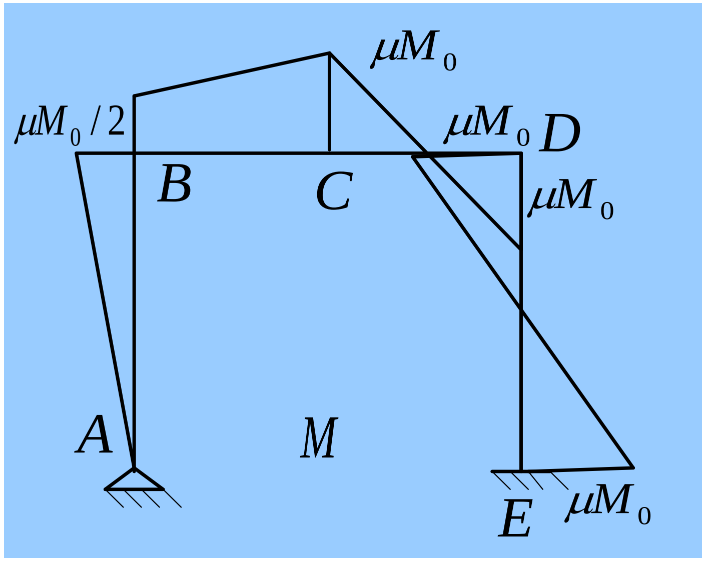
Plastic Limit in Joint B :

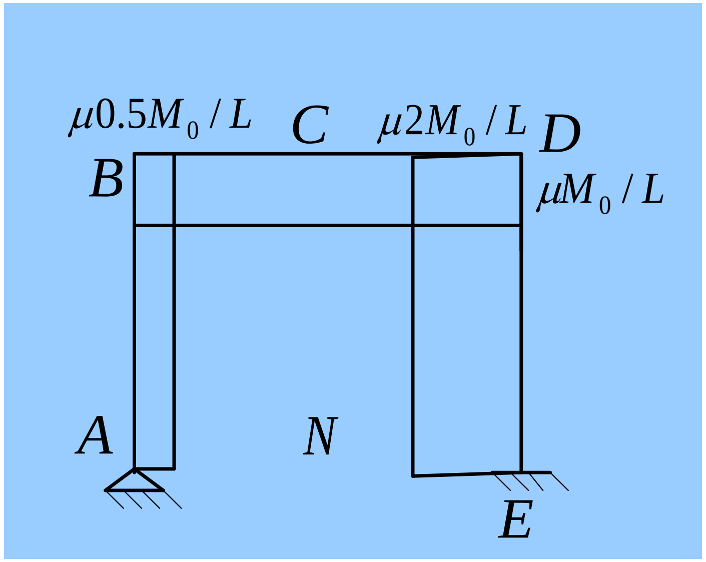
$$\frac{\left|\mu M_0/2\right|}{M_0}+\left(\frac{L\mu M_0/L}{\lambda_0 M_0}\right)^2\leq 1$$

$$2\mu^2+\lambda_0^2\mu-2\lambda_0^2\leq 0$$

$$\mu\leq\frac{\lambda_0}{4}\left(\sqrt{\lambda_0^2+16}-\lambda_0\right)$$

Example, continue…:





Plastic Limit in Joints C, D, E :

$$\frac{|\mu M_0|}{M_0}+\left(\frac{L\mu 2M_0/L}{\lambda_0 M_0}\right)^2\leq 1$$

$$4\mu^2+\lambda_0^2\mu-\lambda_0^2\leq 0$$

$$\mu\leq\frac{\lambda_0}{8}\left(\sqrt{\lambda_0^2+16}-\lambda_0\right)$$

Example, continue…:

Plastic Limit in Joint B :

$$\mu \le \frac{\lambda_0}{4}\left(\sqrt{\lambda_0^2+16}-\lambda_0\right)$$

Plastic Limit in Joints C, D, E :

$$\mu \le \frac{\lambda_0}{8}\left(\sqrt{\lambda_0^2+16}-\lambda_0\right)$$

$$\mu = \frac{\lambda_0}{8}\left(\sqrt{\lambda_0^2+16}-\lambda_0\right)$$

In the Rectangular Section:

$$h_0=\frac{M_0}{N_0}=\frac{\sigma_0 bh^2/4}{\sigma_0 bh}=\frac{h}{4}$$

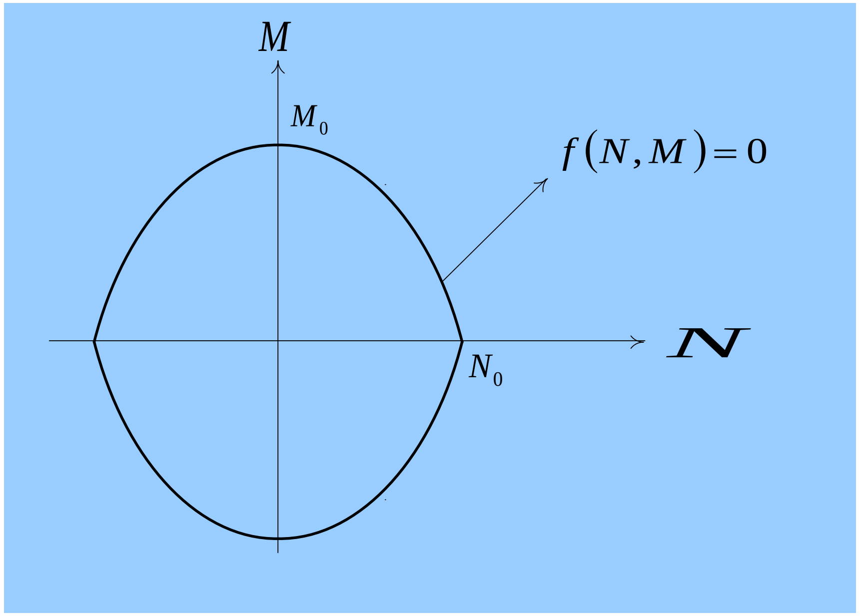
Suppose:

$$\frac{L}{h}=10 \rightarrow h_0=\frac{L}{40} \rightarrow \lambda_0=40 \rightarrow \mu = 0.9975$$

Limit Analysis in Frames, Considering the Effect of Normal Forces:

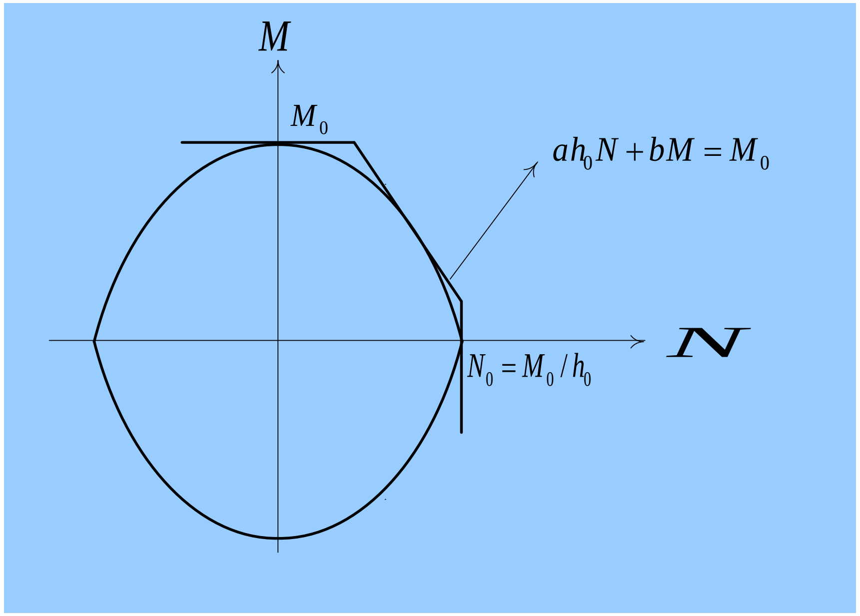
-Actual Solution:

Plastic Limit : $f(N,M)=0$



Linearized Plastically Admissible Condition :

$$\underbrace{\begin{bmatrix} a_1h_0 & b_1 \\ a_2h_0 & b_2 \\ a_3h_0 & b_3 \end{bmatrix}}_{\widehat{\mathbf{Y}}} \underbrace{\begin{Bmatrix} |N| \\ |M| \end{Bmatrix}}_{|\widehat{\mathbf{s}}|} \le \underbrace{\begin{Bmatrix} M_0 \\ M_0 \\ M_0 \end{Bmatrix}}_{\widehat{\mathbf{M}}_0}$$



$$\widehat{\mathbf{Y}}|\widehat{\mathbf{s}}| \le \widehat{\mathbf{M}}_0$$

Limit Analysis in Frames, Considering the Effect of Normal Forces:

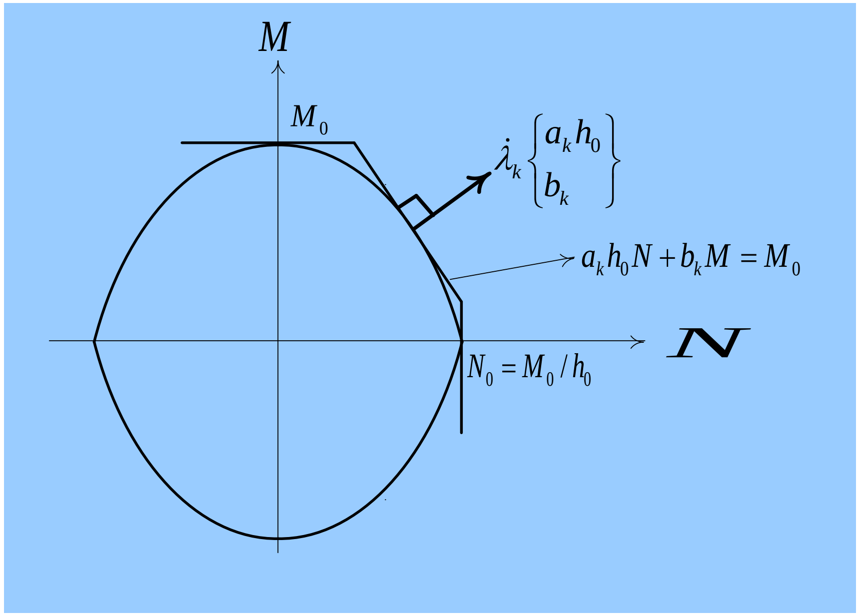
-Actual Solution, continue…:

Suppose the Active Segment of the Plastic Limit Polygon Is k:

$$a_k h_0 N + b_k M = M_0$$

$$\hat{\dot{\mathbf{e}}} = \begin{Bmatrix} \dot{e} \\ \dot{\theta} \end{Bmatrix} = \dot{\lambda}_k \begin{Bmatrix} a_k h_0 \\ b_k \end{Bmatrix} = \begin{Bmatrix} a_k h_0 \\ b_k \end{Bmatrix} \dot{\lambda}_k$$

$$\hat{\dot{\mathbf{e}}} = \begin{Bmatrix} \dot{e} \\ \dot{\theta} \end{Bmatrix} = \underbrace{\begin{bmatrix} a_1 h_0 & a_2 h_0 & a_3 h_0 \\ b_1 & b_2 & b_3 \end{bmatrix}}_{\hat{\mathbf{Y}}^T} \underbrace{\begin{Bmatrix} 0 \\ \dot{\lambda}_k \\ 0 \end{Bmatrix}}_{\hat{\dot{\boldsymbol{\lambda}}}}$$



$$\hat{\dot{\mathbf{e}}} = \hat{\mathbf{Y}}^T \hat{\dot{\boldsymbol{\lambda}}}$$

$$D_{\text{int}} = N\dot{e} + M\dot{\theta}$$

$$D_{\text{int}} = \dot{\lambda}_k \left(a_k h_0 N + b_k M \right)$$

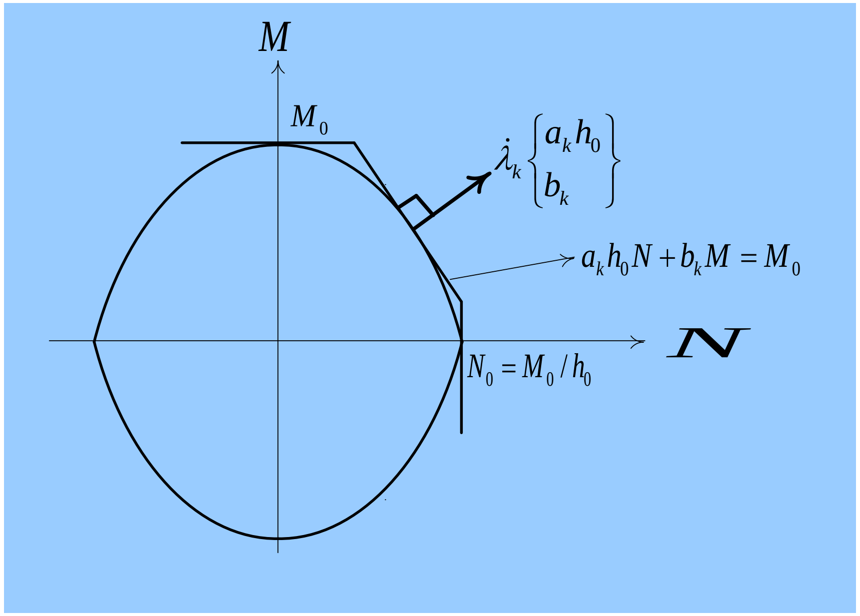
$$D_{\text{int}} = \dot{\lambda}_k M_0 = \hat{\mathbf{M}}_0^T \hat{\dot{\boldsymbol{\lambda}}}$$

Limit Analysis in Frames, Considering the Effect of Normal Forces:

-Actual Solution, continue…:

Suppose the Active Segment of the Plastic Limit Polygon Is k:

$$a_k h_0 N + b_k M = M_0 \qquad \dot{\lambda}_k \geq 0$$



$$i \neq k$$

$$a_i h_0 N + b_i M \neq M_0 \qquad \dot{\lambda}_i = 0$$

$$\dot{\hat{\boldsymbol{\lambda}}} = \begin{Bmatrix} 0 \\ \dot{\lambda}_k \\ 0 \end{Bmatrix}$$

$$\hat{\mathbf{Y}}|\hat{\mathbf{s}}| \leq \hat{\mathbf{M}}_0$$

$$\left(\hat{\mathbf{Y}}|\hat{\mathbf{s}}| - \hat{\mathbf{M}}_0\right)^T \dot{\hat{\boldsymbol{\lambda}}} = 0 \quad \text{:Karush – Kuhn – Tucker Condition}$$

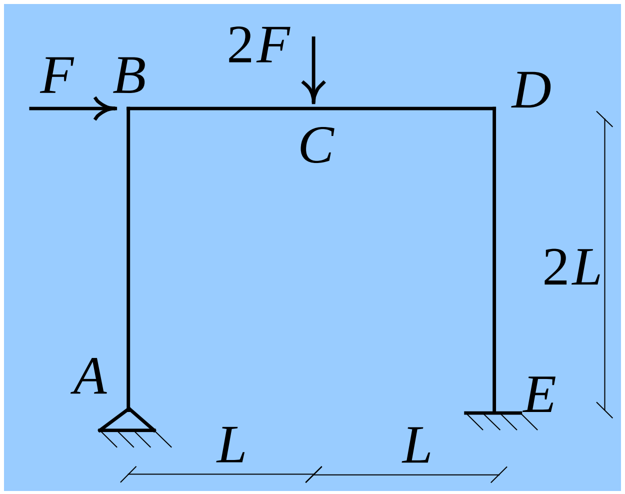
Limit Analysis in Frames, Considering the Effect of Normal Forces:

-Actual Solution, example:

Simple Estimate:

$$\mu = 0.9975$$

$$\frac{1.2469 M_0}{L} < F < \frac{1.2500 M_0}{L}$$

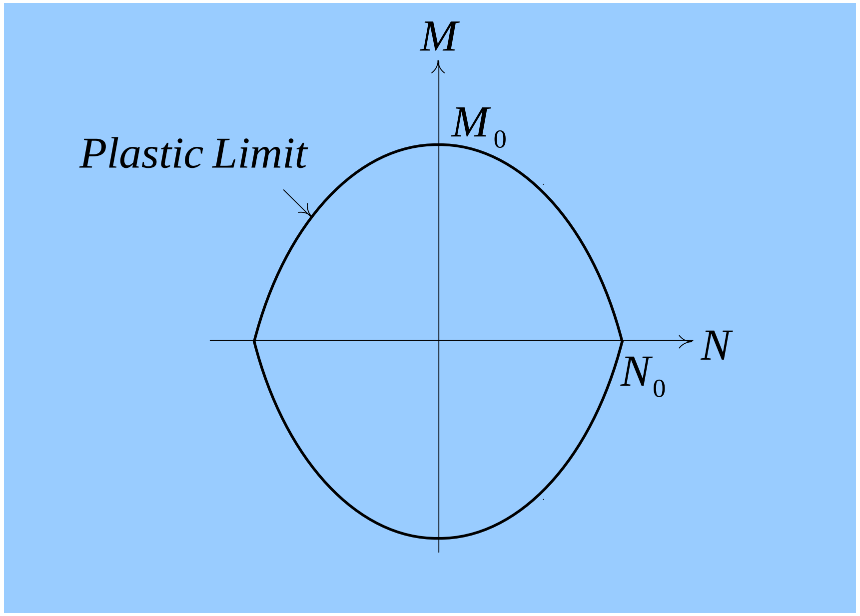


Plastic Limit :

$$\frac{|M|}{M_0} + \left(\frac{N}{N_0}\right)^2 = 1$$

Plastic Deformation:

$$\begin{Bmatrix} \dot{e} \\ \dot{\theta} \end{Bmatrix} = \dot{\lambda} \begin{Bmatrix} 2N / N_0^2 \\ 1/M_0 \dfrac{|M|}{M} \end{Bmatrix}$$



Limit Analysis in Frames, Considering the Effect of Normal Forces:

Example, continue…:

$$\frac{M_0}{N_0} = h_0 = \frac{h}{4} = \frac{L}{40}$$

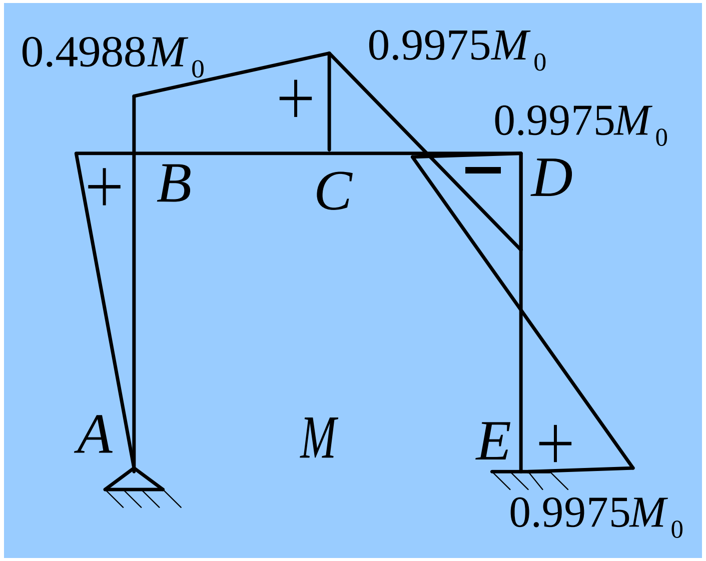
Plastic Limit :

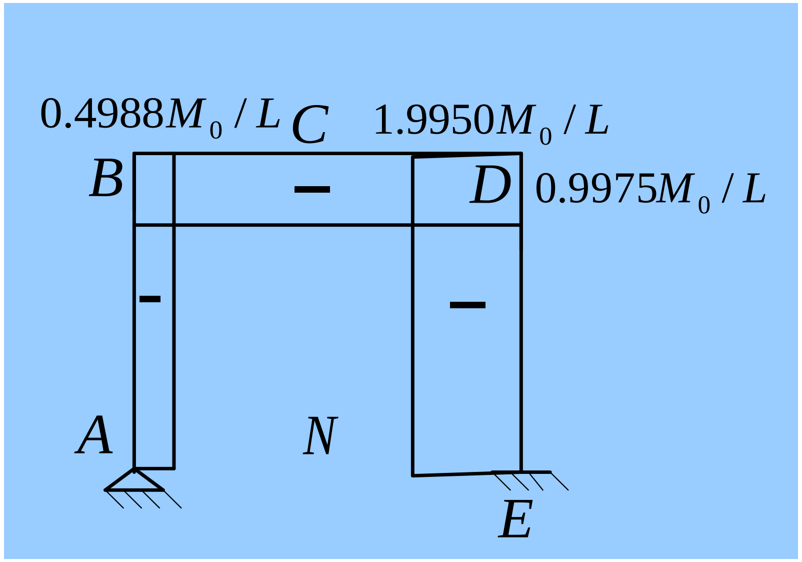
$$|M| + \frac{L^2 N^2}{1600 M_0} = M_0$$

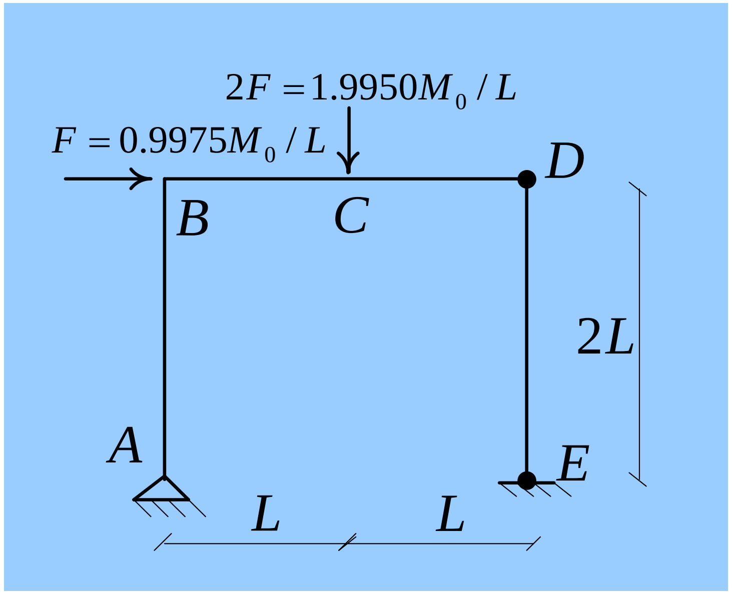
Plastic Deformation:

$$\begin{Bmatrix} \dot{e} \\ \dot{\theta} \end{Bmatrix} = \dot{\lambda} \begin{Bmatrix} \dfrac{L^2 N}{800 M_0} \\ \dfrac{|M|}{M} \end{Bmatrix}$$

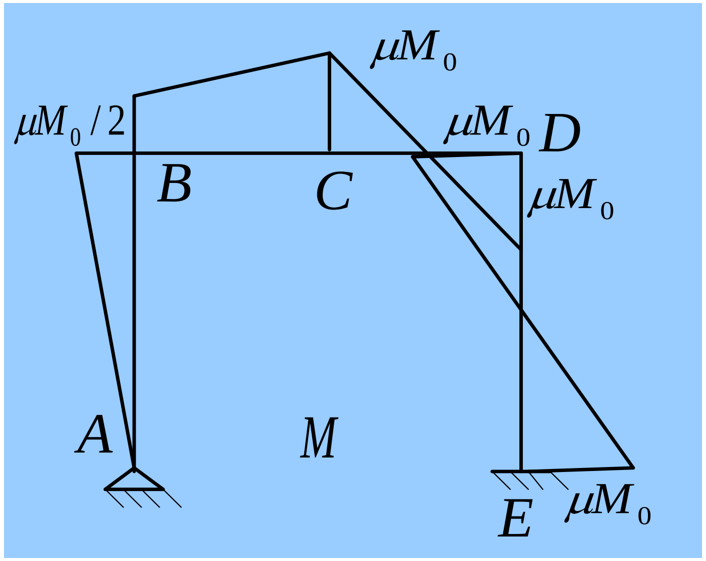
Limit Analysis in Frames, Considering the Effect of Normal Forces: Example, continue…:

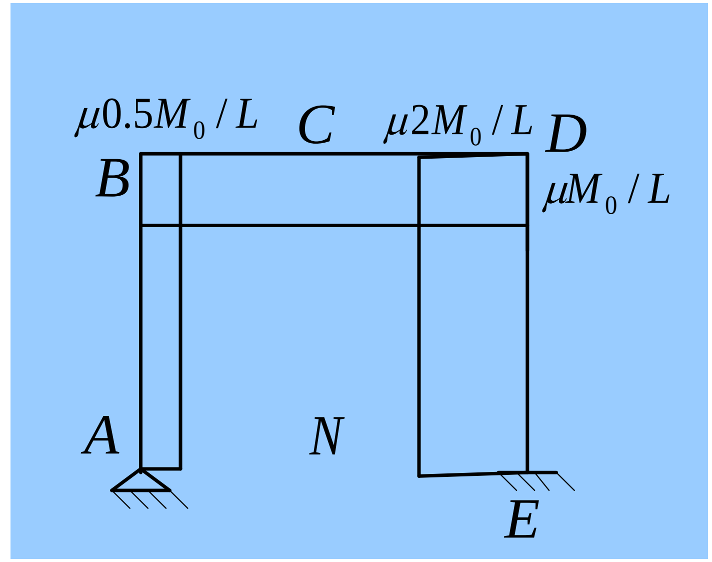






Example, continue…:





Plastic Limit in the Joint C:

$$\frac{|\mu M_0|}{M_0}+\left(\frac{L\mu M_0/L}{\lambda_0 M_0}\right)^2=1$$

$$\mu^2+\lambda_0^2\mu-\lambda_0^2=0$$

$$\mu=\frac{\lambda_0}{2}\left(\sqrt{\lambda_0^2+4}-\lambda_0\right)$$

$$\lambda_0=40\rightarrow\mu=0.9994$$

Limit Analysis in Frames, Considering the Effect of Normal Forces:

Example, continue…:

$$\begin{Bmatrix} N_C \\ M_C \end{Bmatrix} = \begin{Bmatrix} -0.9994\dfrac{M_0}{L} \\ 0.9994M_0 \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{e}_C \\ \dot{\theta}_C \end{Bmatrix} = \dot{\lambda}_C \begin{Bmatrix} -\dfrac{0.9994L}{800} \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} N_D \\ M_D \end{Bmatrix} = \begin{Bmatrix} -1.995\dfrac{M_0}{L} \\ -0.9975M_0 \end{Bmatrix}$$

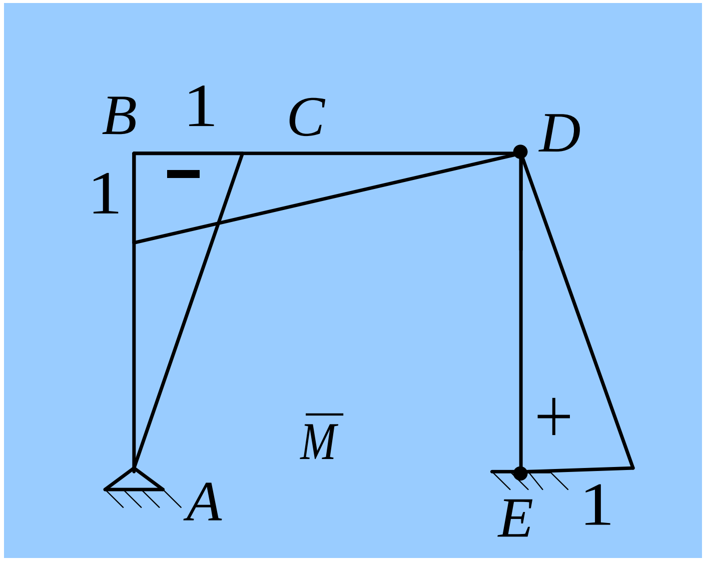
$$\begin{Bmatrix} \dot{e}_D \\ \dot{\theta}_D \end{Bmatrix} = \dot{\lambda}_D \begin{Bmatrix} -\dfrac{1.995L}{800} \\ -1 \end{Bmatrix}$$

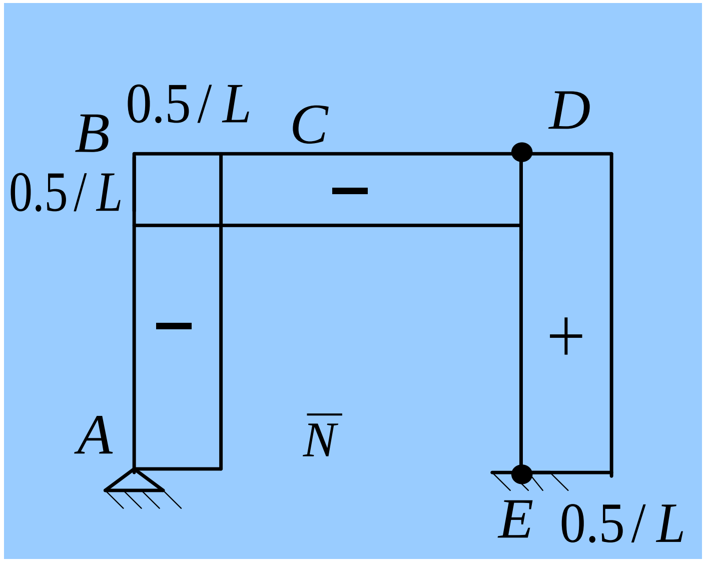
$$\begin{Bmatrix} N_E \\ M_E \end{Bmatrix} = \begin{Bmatrix} -1.995\dfrac{M_0}{L} \\ 0.9975M_0 \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{e}_E \\ \dot{\theta}_E \end{Bmatrix} = \dot{\lambda}_E \begin{Bmatrix} -\dfrac{1.995L}{800} \\ 1 \end{Bmatrix}$$

$$D_{\text{int}} = \dot{\lambda}_C M_0 + \dot{\lambda}_D M_0 + \dot{\lambda}_E M_0$$

Example, continue…: Compatibility Equations :





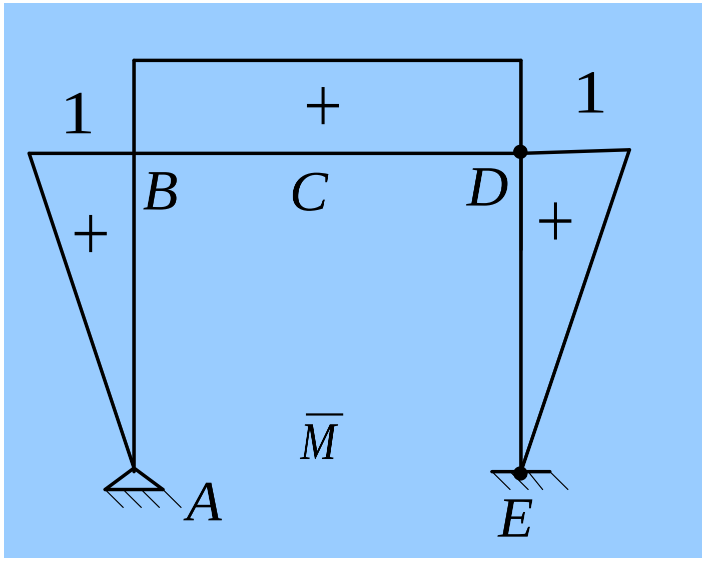
Virtual Work :

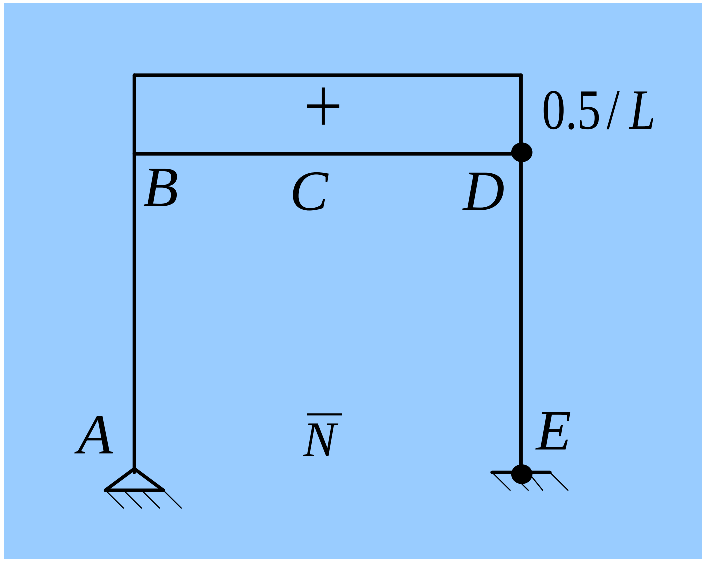
$$1\dot{\theta}_E - 0.5\dot{\theta}_C - \frac{0.5}{L}\dot{e}_C + \frac{0.5}{L}\dot{e}_D + \frac{0.5}{L}\dot{e}_E = 0$$

$$\dot{\lambda}_E - 0.5\dot{\lambda}_C - \frac{0.5}{L}\dot{\lambda}_C\frac{-0.9994L}{800} + \frac{0.5}{L}\dot{\lambda}_D\frac{-1.995L}{800} + \frac{0.5}{L}\dot{\lambda}_E\frac{-1.995L}{800} = 0$$

$$3196.01\dot{\lambda}_E - 1598.0012\dot{\lambda}_C - 3.99\dot{\lambda}_D = 0$$

Example, continue…: Compatibility Equations :





Virtual Work :

$$1\dot{\theta}_C + 1\dot{\theta}_D + \frac{0.5}{L}\dot{e}_C = 0$$

$$\dot{\lambda}_C - \dot{\lambda}_D - \frac{0.5}{L}\dot{\lambda}_C \frac{0.9994L}{800} = 0$$

$$3198.0012\dot{\lambda}_C - 3200\dot{\lambda}_D = 0$$

Example, continue…:

Compatibility Equations :

$$\begin{cases} 3196.01\dot{\lambda}_E - 1598.0012\dot{\lambda}_C - 3.99\dot{\lambda}_D = 0 \\ 3198.0012\dot{\lambda}_C - 3200\dot{\lambda}_D = 0 \end{cases}$$

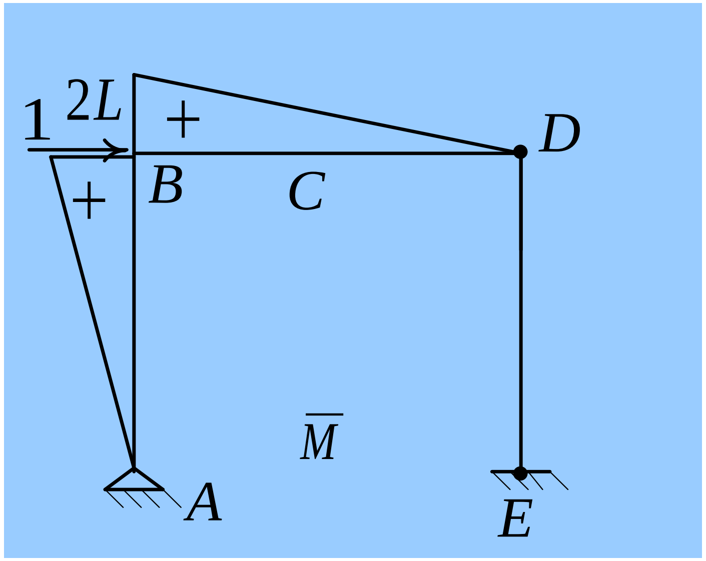
$$\begin{cases} \dot{\lambda}_E = 0.5012\dot{\lambda}_C \\ \dot{\lambda}_D = 0.9994\dot{\lambda}_C \end{cases}$$

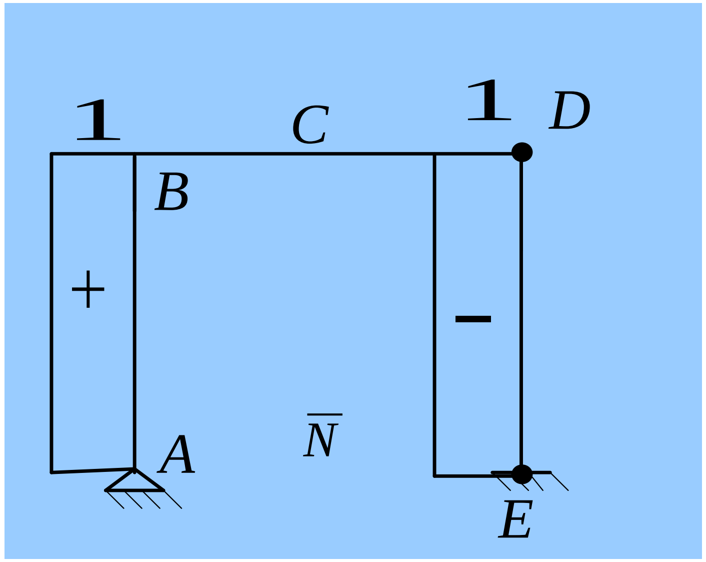
Dissipation Power :

$$D_{\text{int}} = \dot{\lambda}_C M_0 + \dot{\lambda}_D M_0 + \dot{\lambda}_E M_0$$

$$D_{\text{int}} = 2.5006\dot{\lambda}_C M_0$$

Example, continue…: Displacements Definition:





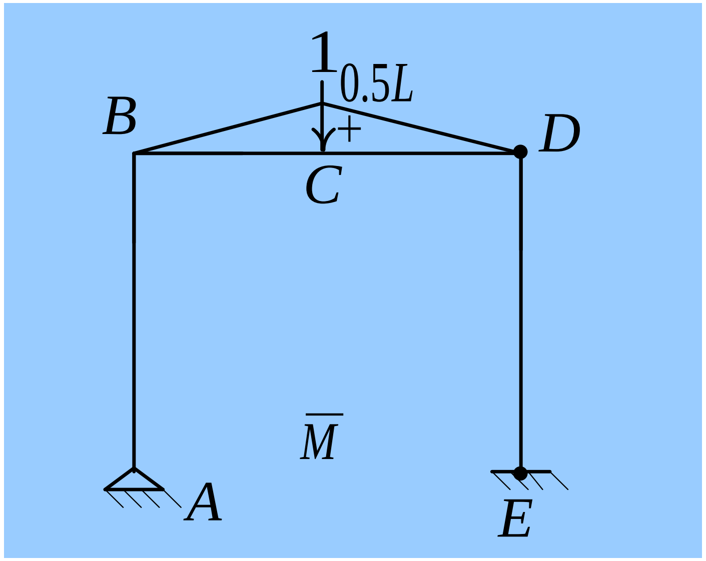
Virtual Work :

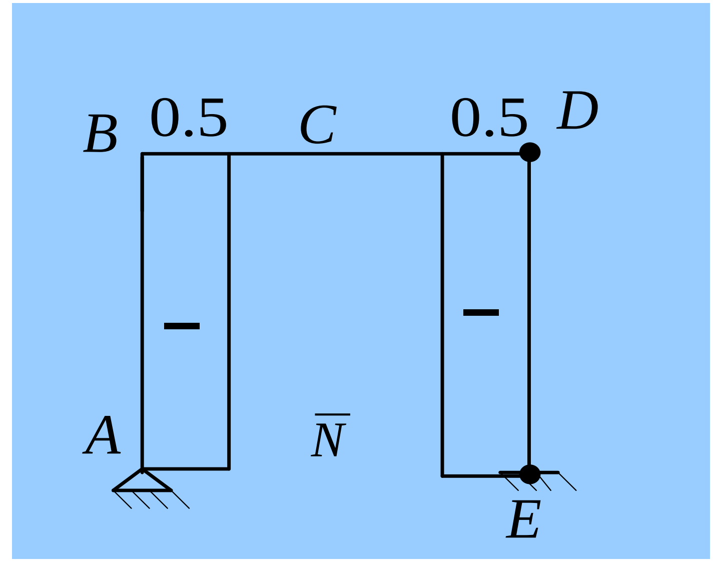
$$1\dot{u}_B = L\dot{\theta}_C - 1\dot{e}_D - 1\dot{e}_E$$

$$\dot{u}_B = L\dot{\lambda}_C - \dot{\lambda}_D \frac{-1.995L}{800} - \dot{\lambda}_E \frac{-1.995L}{800}$$

$$\dot{u}_B = 1.0037\dot{\lambda}_C L$$

Example, continue…: Displacements Definition:





Virtual Work :

$$1\dot{v}_C = 0.5L\dot{\theta}_C - 0.5\dot{e}_D - 0.5\dot{e}_E$$

$$\dot{v}_C = 0.5L\dot{\lambda}_C - 0.5\dot{\lambda}_D \frac{-1.995L}{800} - 0.5\dot{\lambda}_E \frac{-1.995L}{800}$$

$$\dot{v}_C = 0.5019\dot{\lambda}_C L$$

Example, continue…:

$$\dot{W} = F\dot{u}_B + 2F\dot{v}_C$$

$$\dot{W} = 2.0075FL\dot{\lambda}_C$$

Power Equality :

$$D_{\text{int}} = \dot{W}$$

$$2.5006\dot{\lambda}_C M_0 = 2.0075FL\dot{\lambda}_C$$

$$F = 1.2456\frac{M_0}{L}$$

Limit Analysis in Frames, Considering the Effect of Normal Forces:

-Actual Solution of Collapse Load:

Linearized Plastically Admissible Condition For All of The Critical Cross Section:

$$\mathbf{Y}|\mathbf{s}| \le \mathbf{M}_0$$

$$\begin{bmatrix} (a_1h_0)_{ij} & (b_1)_{ij} & 0 & 0 & 0 & 0 \\ (a_2h_0)_{ij} & (b_2)_{ij} & 0 & 0 & 0 & 0 \\ (a_3h_0)_{ij} & (b_3)_{ij} & 0 & 0 & 0 & 0 \\ 0 & 0 & (a_1h_0)_{kl} & (b_1)_{kl} & 0 & 0 \\ 0 & 0 & (a_2h_0)_{kl} & (b_2)_{kl} & 0 & 0 \\ 0 & 0 & (a_3h_0)_{kl} & (b_3)_{kl} & 0 & 0 \\ 0 & 0 & 0 & 0 & (a_1h_0)_{mn} & (b_1)_{mn} \\ 0 & 0 & 0 & 0 & (a_2h_0)_{mn} & (b_2)_{mn} \\ 0 & 0 & 0 & 0 & (a_3h_0)_{mn} & (b_3)_{mn} \end{bmatrix} \begin{Bmatrix} |N_{ij}| \\ |M_{ij}| \\ |N_{kl}| \\ |M_{kl}| \\ |N_{mn}| \\ |M_{mn}| \end{Bmatrix} = \begin{Bmatrix} M_{0ij} \\ M_{0ij} \\ M_{0ij} \\ M_{0kl} \\ M_{0kl} \\ M_{0kl} \\ M_{0mn} \\ M_{0mn} \\ M_{0mn} \end{Bmatrix}$$

Limit Analysis in Frames, Considering the Effect of Normal Forces:

-Actual Solution of Collapse Load:

Plastic Deformation

For All of The Critical Cross Section:

$$\dot{\mathbf{e}} = \mathbf{Y}^T \dot{\boldsymbol{\lambda}}$$

$$\begin{Bmatrix} \dot{e}_{ij} \\ \dot{\theta}_{ij} \\ \dot{e}_{kl} \\ \dot{\theta}_{kl} \\ \dot{e}_{mn} \\ \dot{\theta}_{mn} \end{Bmatrix} = \begin{bmatrix} (a_1h_0)_{ij} & (b_1)_{ij} & 0 & 0 & 0 & 0 \\ (a_2h_0)_{ij} & (b_2)_{ij} & 0 & 0 & 0 & 0 \\ (a_3h_0)_{ij} & (b_3)_{ij} & 0 & 0 & 0 & 0 \\ 0 & 0 & (a_1h_0)_{kl} & (b_1)_{kl} & 0 & 0 \\ 0 & 0 & (a_2h_0)_{kl} & (b_2)_{kl} & 0 & 0 \\ 0 & 0 & (a_3h_0)_{kl} & (b_3)_{kl} & 0 & 0 \\ 0 & 0 & 0 & 0 & (a_1h_0)_{mn} & (b_1)_{mn} \\ 0 & 0 & 0 & 0 & (a_2h_0)_{mn} & (b_2)_{mn} \\ 0 & 0 & 0 & 0 & (a_3h_0)_{mn} & (b_3)_{mn} \end{bmatrix}^T \begin{Bmatrix} \dot{\lambda}_{1ij} \\ \dot{\lambda}_{2ij} \\ \dot{\lambda}_{3ij} \\ \dot{\lambda}_{1kl} \\ \dot{\lambda}_{2kl} \\ \dot{\lambda}_{3kl} \\ \dot{\lambda}_{1mn} \\ \dot{\lambda}_{2mn} \\ \dot{\lambda}_{3mn} \end{Bmatrix}$$

Limit Analysis in Frames, Considering the Effect of Normal Forces:

-Actual Solution of Collapse Load:

Dissipation Power
For All of The Critical Cross Section:

$$D_{\text{int}} = \mathbf{M}_0^T \dot{\boldsymbol{\lambda}}$$

$$D_{\text{int}} = \begin{bmatrix} M_{0ij} & M_{0ij} & M_{0ij} & M_{0kl} & M_{0kl} & M_{0kl} & M_{0mn} & M_{0mn} & M_{0mn} \end{bmatrix} \begin{Bmatrix} \dot{\lambda}_{1ij} \\ \dot{\lambda}_{2ij} \\ \dot{\lambda}_{3ij} \\ \dot{\lambda}_{1kl} \\ \dot{\lambda}_{2kl} \\ \dot{\lambda}_{3kl} \\ \dot{\lambda}_{1mn} \\ \dot{\lambda}_{2mn} \\ \dot{\lambda}_{3mn} \end{Bmatrix}$$

Limit Analysis in Frames, Considering The Effect of Normal Forces:

-Actual Solution of Collapse Load:

Karush – Kuhn – Tucker Condition
For All of The Critical Cross Section:

$$\left(\mathbf{Y}\left|\mathbf{s}\right|-\mathbf{M}_0\right)^T\dot{\boldsymbol{\lambda}}=0$$

$$\left(\begin{bmatrix}
(a_1h_0)_{ij} & (b_1)_{ij} & 0 & 0 & 0 & 0\\
(a_2h_0)_{ij} & (b_2)_{ij} & 0 & 0 & 0 & 0\\
(a_3h_0)_{ij} & (b_3)_{ij} & 0 & 0 & 0 & 0\\
0 & 0 & (a_1h_0)_{kl} & (b_1)_{kl} & 0 & 0\\
0 & 0 & (a_2h_0)_{kl} & (b_2)_{kl} & 0 & 0\\
0 & 0 & (a_3h_0)_{kl} & (b_3)_{kl} & 0 & 0\\
0 & 0 & 0 & 0 & (a_1h_0)_{mn} & (b_1)_{mn}\\
0 & 0 & 0 & 0 & (a_2h_0)_{mn} & (b_2)_{mn}\\
0 & 0 & 0 & 0 & (a_3h_0)_{mn} & (b_3)_{mn}
\end{bmatrix}\begin{Bmatrix}\left|N_{ij}\right|\\ \left|M_{ij}\right|\\ \left|N_{kl}\right|\\ \left|M_{kl}\right|\\ \left|N_{mn}\right|\\ \left|M_{mn}\right|\end{Bmatrix}-\begin{Bmatrix}M_{0ij}\\ M_{0ij}\\ M_{0ij}\\ M_{0kl}\\ M_{0kl}\\ M_{0kl}\\ M_{0mn}\\ M_{0mn}\\ M_{0mn}\end{Bmatrix}\right)^T\begin{Bmatrix}\dot{\lambda}_{1ij}\\ \dot{\lambda}_{2ij}\\ \dot{\lambda}_{3ij}\\ \dot{\lambda}_{1kl}\\ \dot{\lambda}_{2kl}\\ \dot{\lambda}_{3kl}\\ \dot{\lambda}_{1mn}\\ \dot{\lambda}_{2mn}\\ \dot{\lambda}_{3mn}\end{Bmatrix}=0$$

Linear Programming in Limit Analysis Considering The Effect of Normal Forces Using Static Approach :

$-\, Equilibrium \quad Equation:$ $$\mu_s \bar{\mathbf{f}} = \mathbf{B}^T \mathbf{s}$$

$-\, Plastically\ Admissible\ Condition:$ $$\mathbf{Y}|\mathbf{s}| \leq \mathbf{M}_0$$

$-\, Lower\ Bound\ Theorm:$ $$Maximize\ \mu_s$$

Linear Programming in Limit Analysis Considering The Effect of Normal Forces Using Static Approach :

$-\,Equilibrium \quad Equation:$

$$\mu_s \bar{\mathbf{f}} = \mathbf{B}^T \mathbf{s}$$

$-\,Plastically\ Admissible\ Condition:$

$$\mathbf{Y}|\mathbf{s}| \leq \mathbf{M}_0$$

$-\,Lower\ Bound\ Theorm:$

$$Maximize\ \mu_s$$

$$\left.\begin{aligned} \mathbf{s}^+ &= \frac{|\mathbf{s}| - \mathbf{s}}{2} \geq 0 \\ \mathbf{s}^- &= \frac{|\mathbf{s}| - \mathbf{s}}{2} \geq 0 \end{aligned}\right\} \longrightarrow \left\{\begin{aligned} \mathbf{s} &= \mathbf{s}^+ - \mathbf{s}^- \\ |\mathbf{s}| &= \mathbf{s}^+ + \mathbf{s}^- \end{aligned}\right.$$

$$\Delta\mathbf{M} = \mathbf{M}_0 - \mathbf{Y}|\mathbf{s}| \geq 0$$

Linear Programming in Limit Analysis
Considering The Effect of Normal Forces Using Static Approach :

$$Minimize: -\mu_s$$

$$\mathbf{B}^T\mathbf{s}^+ - \mathbf{B}^T\mathbf{s}^- - \mu_s\bar{\mathbf{f}} = \mathbf{0}$$

$$\mathbf{Y}\mathbf{s}^+ + \mathbf{Y}\mathbf{s}^- + \Delta\mathbf{M} = \mathbf{M}_0$$

$$\mathbf{s}^+ \geq \mathbf{0}$$

$$\mathbf{s}^- \geq \mathbf{0}$$

$$\Delta\mathbf{M} \geq \mathbf{0}$$

$$\mu_s \geq 0$$

Linear Programming in Limit Analysis Considering The Effect of Normal Forces Using Kinematic Approach:

$-\,Kinematic\ Equation:$ $\dot{\mathbf{e}} = \mathbf{B}\dot{\mathbf{d}}$

$-\,Normalization:$ $\overline{\mathbf{f}}^{T}\dot{\mathbf{d}} = 1$

$-\,Krush\ Kuhn\ Tucker\ Condition:$ $\left(\mathbf{Y}|\mathbf{s}| - \mathbf{M}_0\right)^T \dot{\boldsymbol{\lambda}} = 0$

$-\,Upper\ Bound\ Theorm:$ $Minimize\ \mu_k = D_{\text{int}} = \mathbf{M}_0^T \dot{\boldsymbol{\lambda}}$

Linear Programming in Limit Analysis Using Kinematic Approach :

$$\left\{\begin{array}{l} Minimize\ \mu_k = \mathbf{M}_0^T \dot{\boldsymbol{\lambda}} \\ \dot{\mathbf{e}} = \mathbf{B}\dot{\mathbf{d}} \\ \bar{\mathbf{f}}^T \dot{\mathbf{d}} = 1 \\ \left(\mathbf{Y}|\mathbf{s}| - \mathbf{M}_0\right)^T \dot{\boldsymbol{\lambda}} = 0 \end{array}\right.$$

$$\left\{\begin{array}{l} \mathbf{P}^T \dot{\mathbf{e}} = \dot{\mathbf{d}} \\ \mathbf{S}^T \dot{\mathbf{e}} = \mathbf{0} \end{array}\right.$$

$$\dot{\mathbf{e}} = \mathbf{Y}^T \dot{\boldsymbol{\lambda}}$$

$$\left\{\begin{array}{l} Minimize\ \mu_k = \mathbf{M}_0^T \dot{\boldsymbol{\lambda}} \\ \mathbf{S}^T \mathbf{Y}^T \dot{\boldsymbol{\lambda}} = \mathbf{0} \\ \bar{\mathbf{f}}^T \mathbf{P}^T \mathbf{Y}^T \dot{\boldsymbol{\lambda}} = 1 \\ \left(\mathbf{Y}\mathbf{s}^+ - \mathbf{Y}\mathbf{s}^- - \mathbf{M}_0\right)^T \dot{\boldsymbol{\lambda}} = 0 \end{array}\right.$$