Plastic Analysis and Design of Structures

Part 1-1

S. Erfani

References:

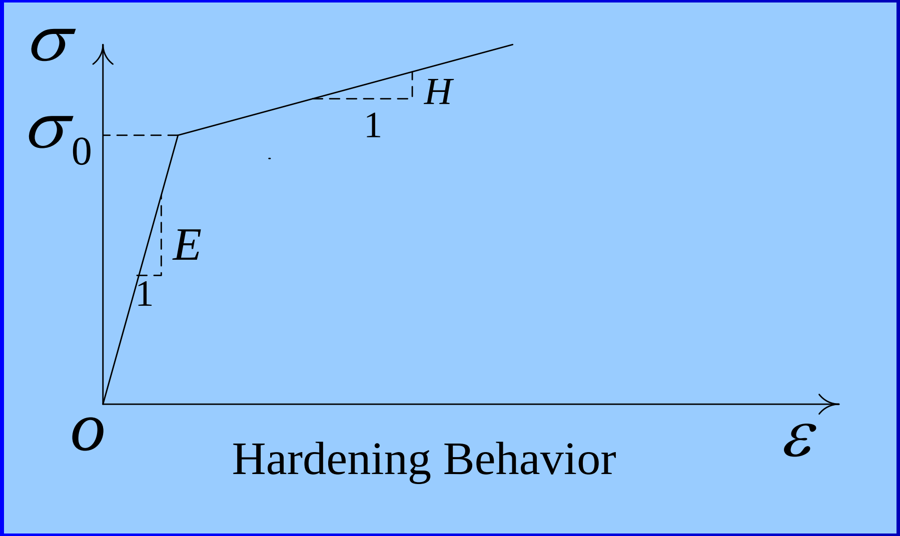
1- M. Jirasek, Z. P. Bazant, Inelastic Analysis of Structures.

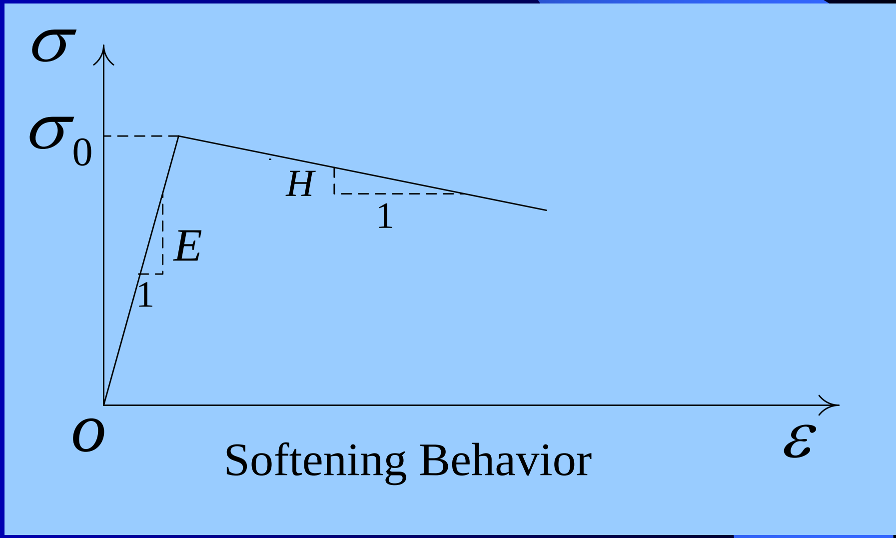
2- M. B. Wong, Plastic Analysis and Design of Steel Structures.

3- O. C. Zienkiewicz, The Finite Element Method.

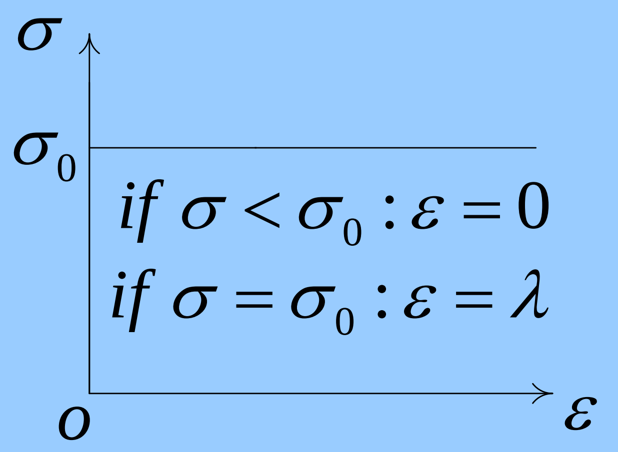
4- AISC 2010, Specification for Structural Steel Building.

The Three Types of Inelastic Behavior:

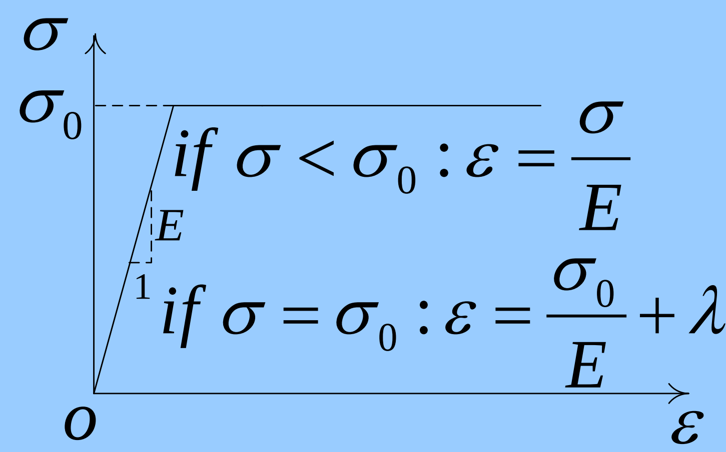




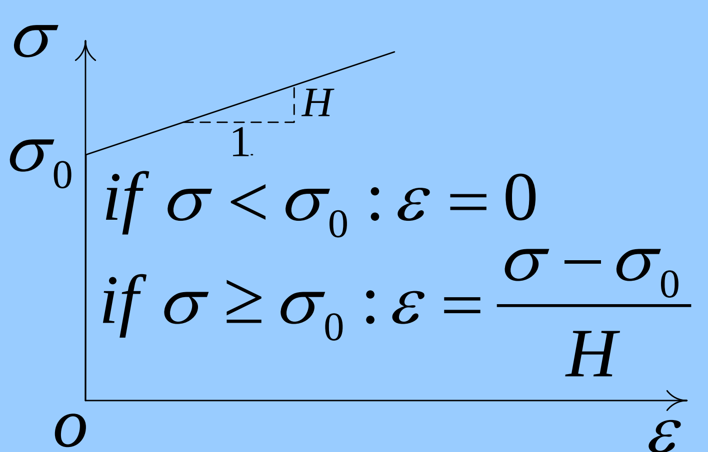
The Ideal Stress Strain Relations:



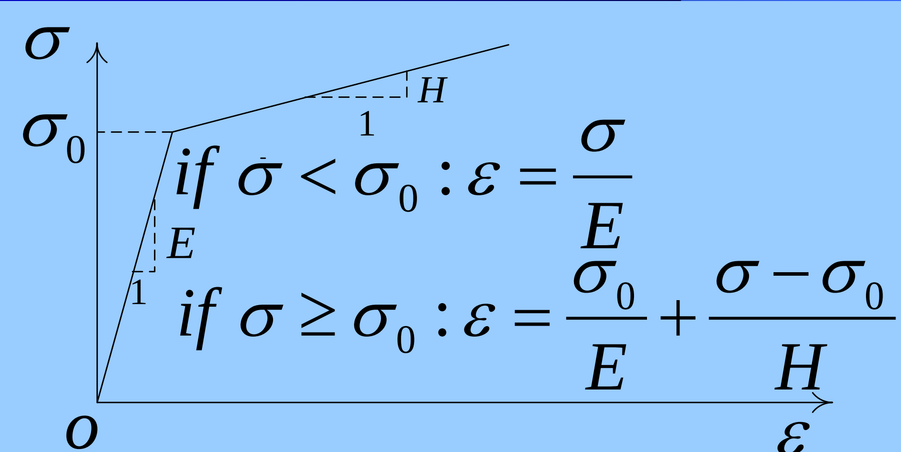
Rigid-Perfectly Plastic



Elastic-Perfectly Plastic



Rigid- Plastic



Elastic-Plastic

Basic Relations:

$$d\varepsilon = d\varepsilon_e + d\varepsilon_p$$

$$\dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_p$$

$$d\varepsilon_e = \frac{d\sigma}{E}$$

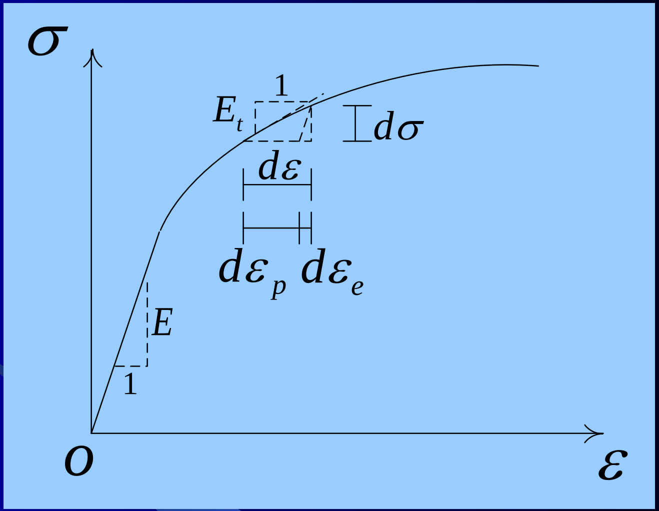
E : *Elastic Modulus*

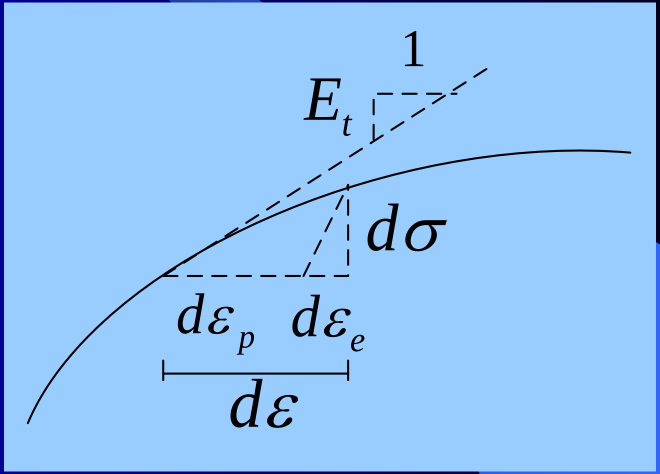
$$d\varepsilon_p = \frac{d\sigma}{E_p}$$

E_p : *Plastic Modulus*

$$d\varepsilon = \frac{d\sigma}{E_t}$$

E_t : *Tangential Modulus*





$$\frac{d\sigma}{E_t} = \frac{d\sigma}{E} + \frac{d\sigma}{E_p} \rightarrow \frac{1}{E_t} = \frac{1}{E} + \frac{1}{E_p}$$

$$\begin{cases} \dfrac{1}{E_t} = \dfrac{1}{E} + \dfrac{1}{E_p} \\[2ex] \dfrac{1}{E_p} = \dfrac{1}{E_t} - \dfrac{1}{E} \end{cases}$$

$$\begin{cases} E_t = \dfrac{EE_p}{E + E_p} \\[2ex] E_p = \dfrac{EE_t}{E - E_t} \end{cases}$$

Basic Relations (continue …):

$$dU = \sigma d\varepsilon : Rate\ of\ \ Energy\ Density$$

$$\dot{U} = \sigma\dot{\varepsilon}$$

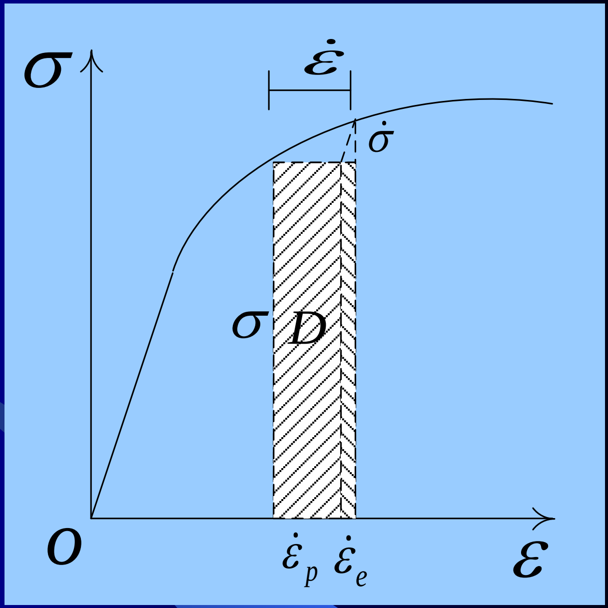
$$\dot{U} = \sigma\dot{\varepsilon}_e + \sigma\dot{\varepsilon}_p$$

$$\sigma\dot{\varepsilon}_e : Rate\ of\ \ Stored\ Elastic\ Energy\ Density$$

$$\sigma\dot{\varepsilon}_p : Rate\ of\ \ Dissipated\ Plastic\ Energy\ Density$$

$$D = \sigma\dot{\varepsilon}_p = D\left(\sigma, \dot{\varepsilon}_p\right): Dissipation\ Power\ Density$$

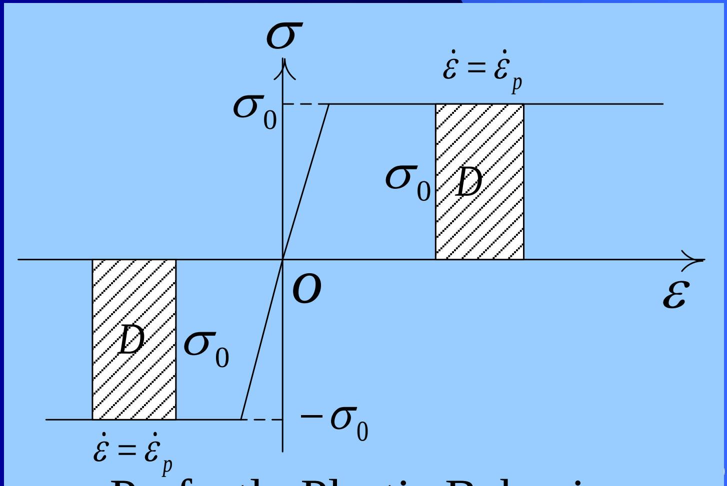
$$Elastic\ Region : D = 0$$



Perfectly Plastic Materials:

$$Elastic\ Region : D = 0$$

$$Plastic\ Region : D = \sigma\dot{\varepsilon}_p = \sigma_0\left|\dot{\varepsilon}_p\right| = D\left(\dot{\varepsilon}_p\right)$$



Elastic Perfectly Plastic Bars:

$L: Bar\ Length$

$A: Cross\ Section\ Area$

$S: Axial\ Force$ $\qquad \sigma A = S$

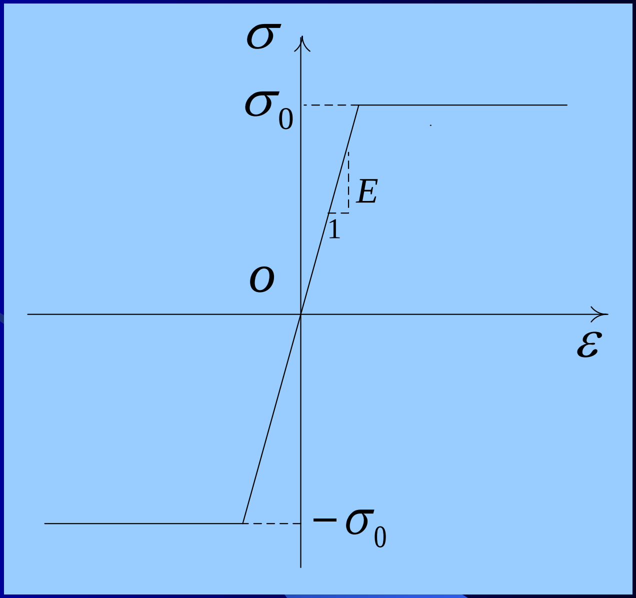
$S_0: Plastic\ Axial\ Force$ $\qquad \sigma_0 A = S_0$

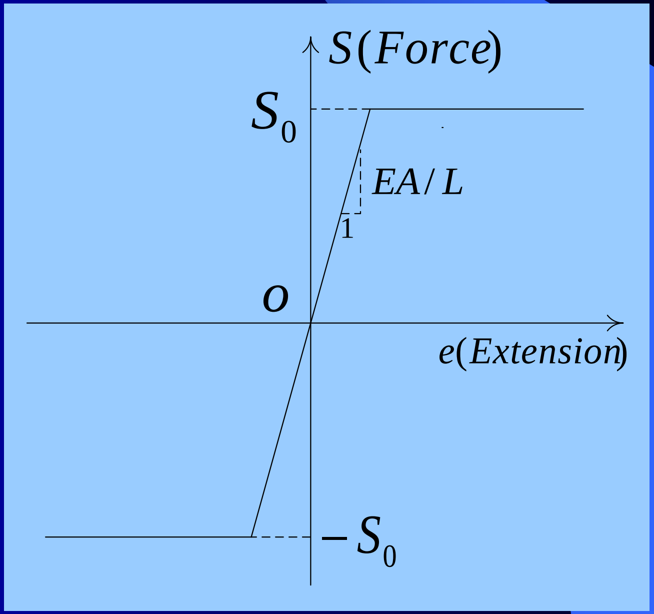
$e: Extention(Change\ of\ \ Length)$

$$if\ |S| < S_0 \rightarrow e = \frac{L}{EA} S : Elastic\ Region$$

$$if\ |S| = S_0 \rightarrow e = Sign(S)\left(\frac{L}{EA} S_0 + \lambda\right) \quad ; \lambda > 0$$

$: Plastic\ Region$





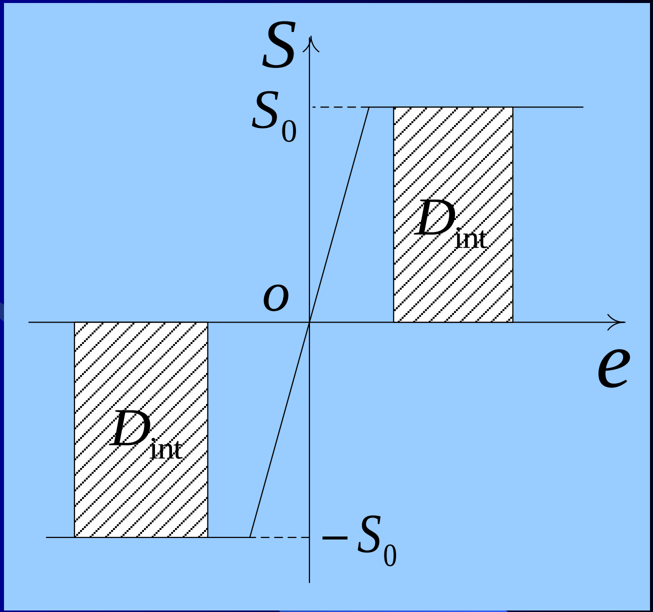
Elastic Perfectly Plastic Bars (continue…):

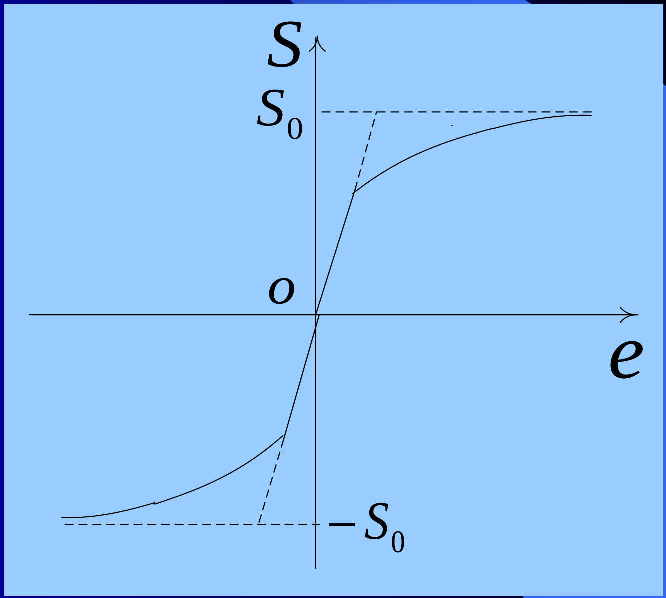
-Dissipation Power:

$$D_{\text{int}} = \int_V D\,dV = \int_V \sigma \dot{\varepsilon}_p\,dV = V\sigma\dot{\varepsilon}_p = A\sigma L\dot{\varepsilon}_p = S\dot{e}_p$$

$$Elastic\ Region: D_{\text{int}} = 0$$

$$Plastic\ Region: D_{\text{int}} = S\dot{e}_p = S_0\left|\dot{e}_p\right|$$





-Residual Stress Effect:

Moment-Curvature Relation in Flexural Beams:

-Bending Usual Assumption (Bernoulli Assumption):
“The Plane Cross Sections Remain Plain and Normal to The Deflected Middle Axis of The Beam”.

$z: \mathit{The\ Fiber\ Depth\ Coordinate\ Measured\ From\ The\ Middle\ Axis}$

$\kappa: \mathit{The\ Middle\ Axis\ Curvature}$

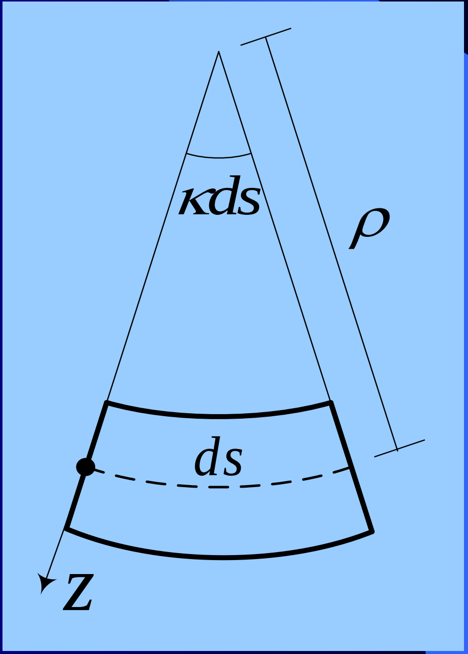
$$\kappa = \frac{\varepsilon}{z}$$

$$\varepsilon = z\kappa$$

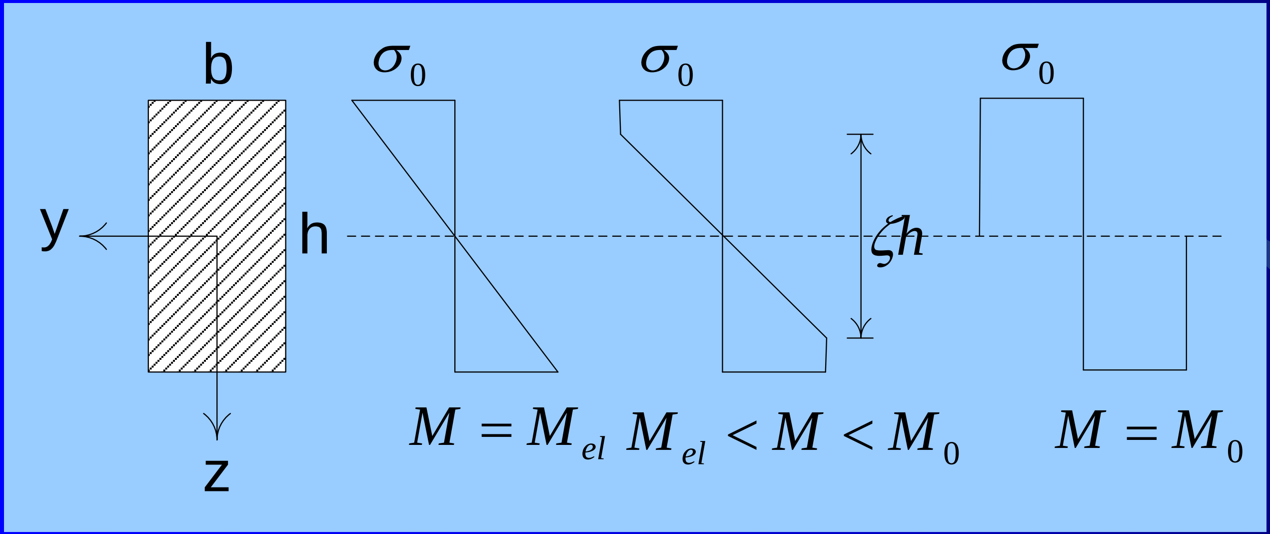
$\rho: \mathit{Radius\ of\ Curvature}$

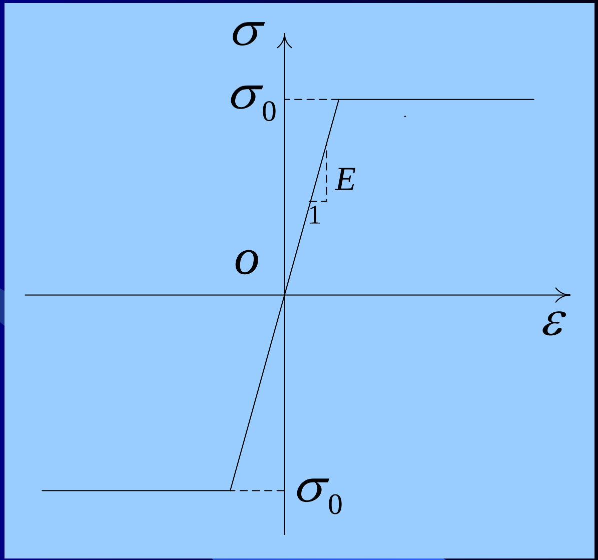
$$\rho = \frac{1}{\kappa}$$

$$M = \int_A \sigma(z) z dA$$



Example: Rectangular Cross Section With Elastic Perfectly Plastic Material:





$M<M_{el}$:

$$M = \int_A z\sigma(z)dA = \int_A zE\varepsilon(z)dA$$

$$M = \int_A zEz\kappa dA = E\kappa\int_A z^2 dA$$

$$I = \int_A z^2 dA$$

$$M = EI\kappa$$

$M=M_{el}$:

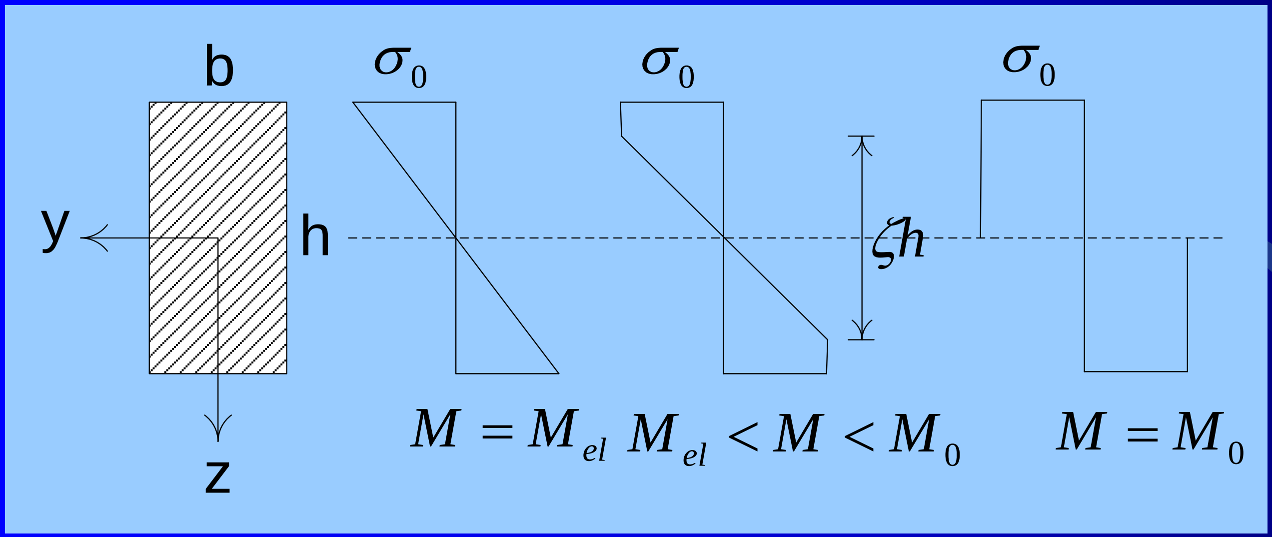
$$M_{el} = EI\kappa_{el}$$

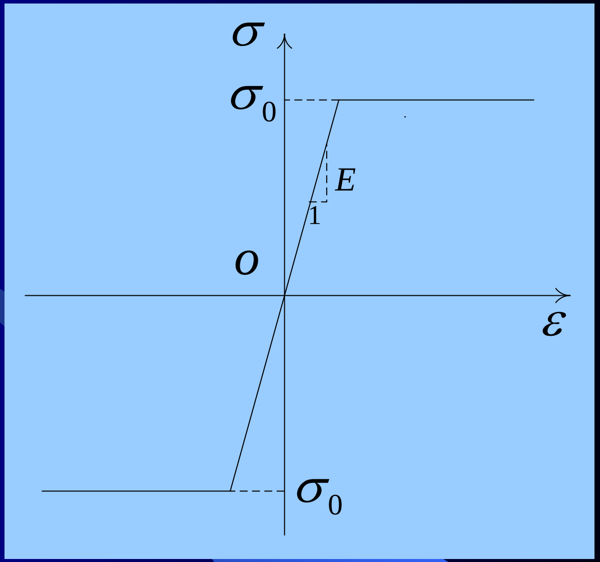
$$\kappa_{el} = \frac{\varepsilon_0}{h/2} = \frac{\sigma_0 / E}{h/2} = \frac{2\sigma_0}{Eh}$$

$$I = \frac{bh^3}{12}$$

$$M_{el} = \sigma_0 \underbrace{\frac{bh^2}{6}}_{W_{el}} = \sigma_0 W_{el}$$

Example: Rectangular Cross Section With Elastic Perfectly Plastic Material (continue …):





$M=M_0$:

$$M_0 = \sigma_0 \frac{bh}{2}\frac{h}{2} = \sigma_0 \underbrace{\frac{bh^2}{4}}_{W_0} = \sigma_0 W_0$$

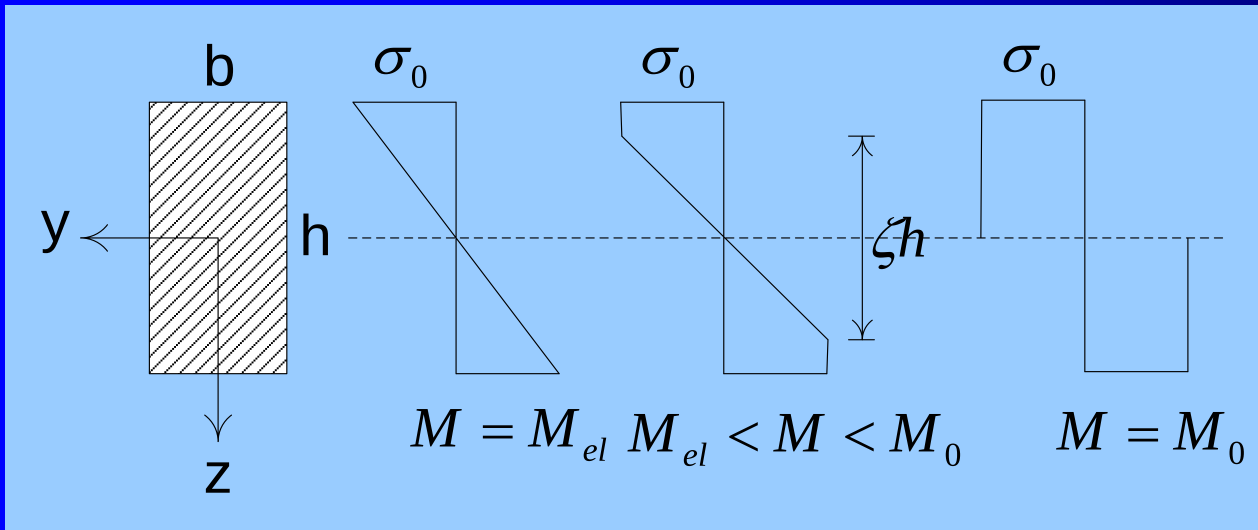
$$\alpha = \frac{W_0}{W_{el}} = \frac{bh^2/4}{bh^2/6} = 1.5 : Shape\ Factor$$

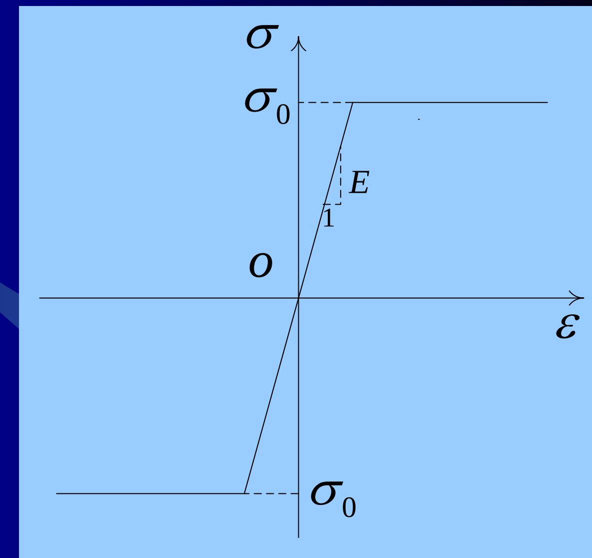
$M_{el}<M<M_0$:

$$M = M_0 - \sigma_0 \frac{b}{2}\frac{\zeta h}{2}\frac{\zeta h}{3} = M_0 - \zeta^2 \sigma_0 \frac{bh^2}{12}$$

$$M = M_0\left(1 - \frac{\zeta^2}{3}\right)$$

Example: Rectangular Cross Section With Elastic Perfectly Plastic Material (continue …):





$M_{el} < M < M_0$:

$$M = M_0\left(1 - \frac{\zeta^2}{3}\right)$$

$$\zeta \frac{h}{2}\kappa = \varepsilon_0 = \frac{\sigma_0}{E}$$

$$\zeta\kappa = \frac{2\sigma_0}{Eh} = \kappa_{el}$$

$$\zeta = \frac{\kappa_{el}}{\kappa}$$

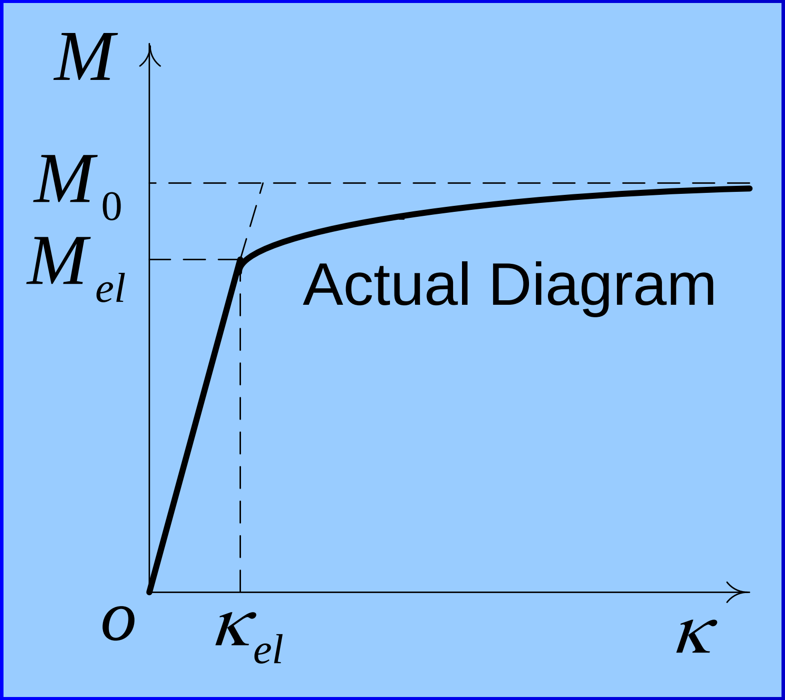
$$M = M_0\left(1 - \frac{\kappa_{el}^2}{3\kappa^2}\right)$$

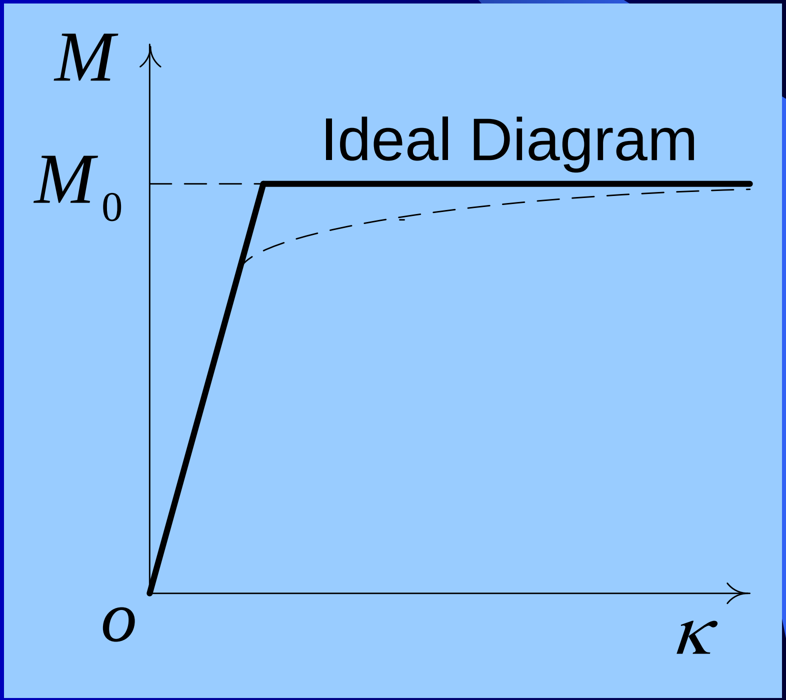
$$\kappa = \frac{\kappa_{el}}{\sqrt{3\left(1 - \frac{M}{M_0}\right)}}$$

$$\kappa \to \infty \equiv M \to M_0$$

Example: Rectangular Cross Section With Elastic Perfectly Plastic Material (continue …):

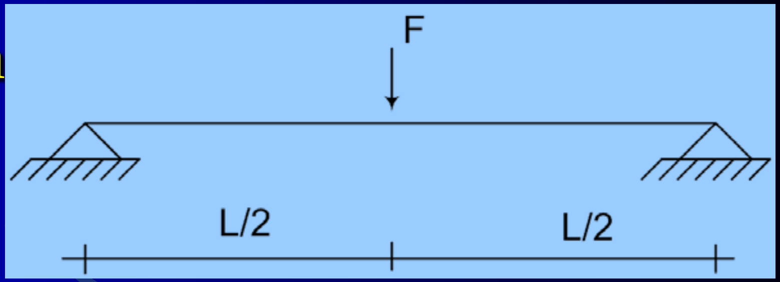
$$M = M_0\left(1 - \frac{\kappa_{el}^{\ 2}}{3\kappa^2}\right)$$



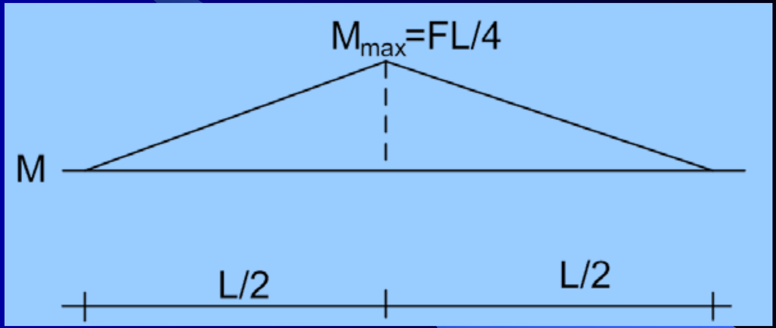


Expanded Plastic zone:

Example) Rectangular Cross Section Beam With Elastic Perfectly Plastic Material:



$$M(x) = \frac{1}{2} F x, \quad x \in \left[0, \frac{L}{2}\right]$$

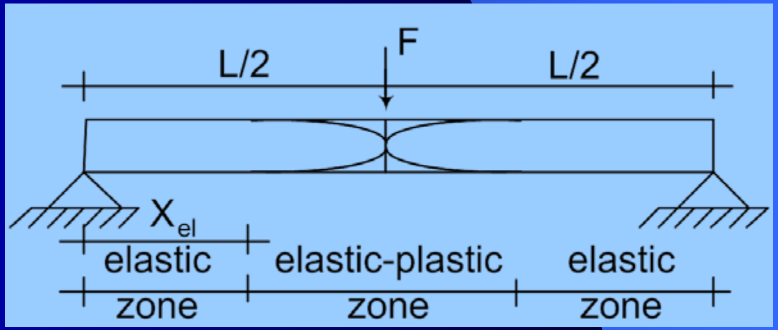


$$x = \frac{L}{2} \rightarrow M_{\max} = \frac{FL}{4}$$

$$F = \frac{4M_{\max}}{L}$$

$$F_{el} = \frac{4M_{el}}{L}$$

$$F_0 = \frac{4M_0}{L}$$



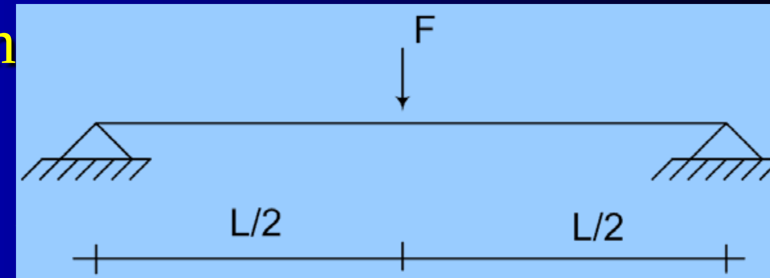
$$M(x) = \frac{2M_{\max}}{L} x, \quad x \in \left[0, \frac{L}{2}\right]$$

$$x = x_{el} \rightarrow M = M_{el} = \frac{M_0}{\alpha} = \frac{2}{3} M_0$$

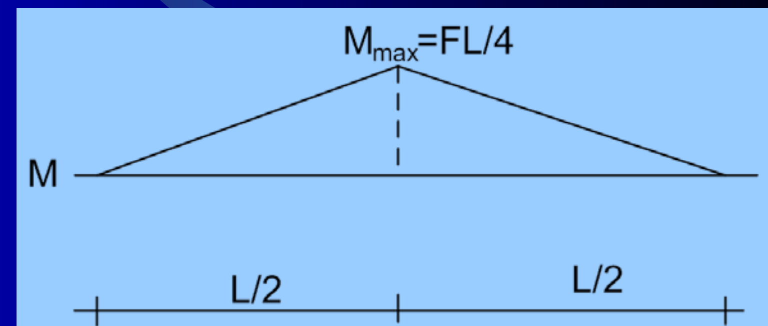
$$x_{el} = \frac{4M_0}{3F}$$

$$\left[x_{el}\right]_{ult} = \frac{L}{3}$$

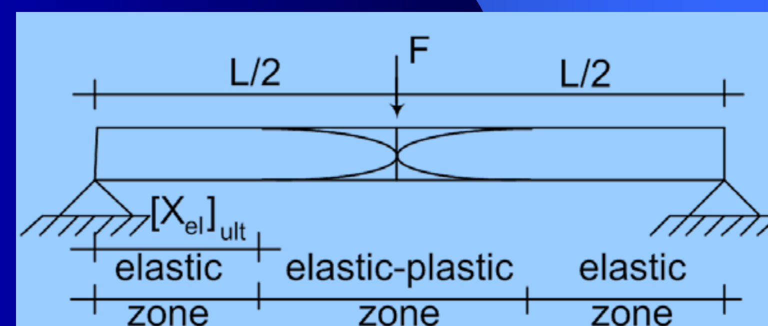
Expanded Plastic zone:
Example) Rectangular Cross Section Beam With Elastic Perfectly Plastic Material (continue …):



$$x \in \left[0, \frac{4M_0}{3F}\right] : Elastic\ Region$$



$$x \in \left[\frac{4M_0}{3F}, \frac{L}{2}\right] : Elastic - Plastic\ Region$$



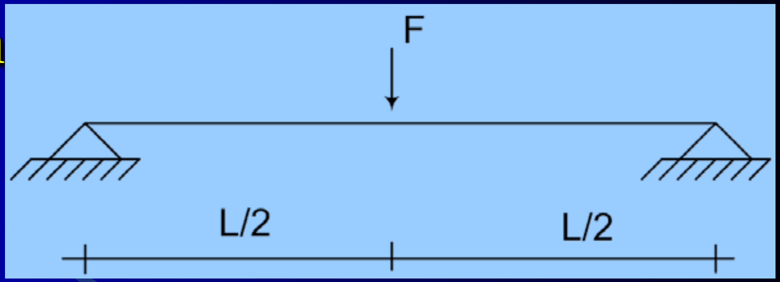
$$x \in \left[\frac{4M_0}{3F}, \frac{L}{2}\right] : M = M_0\left(1 - \frac{\zeta^2}{3}\right) = \frac{2M_{\max}}{L}x$$

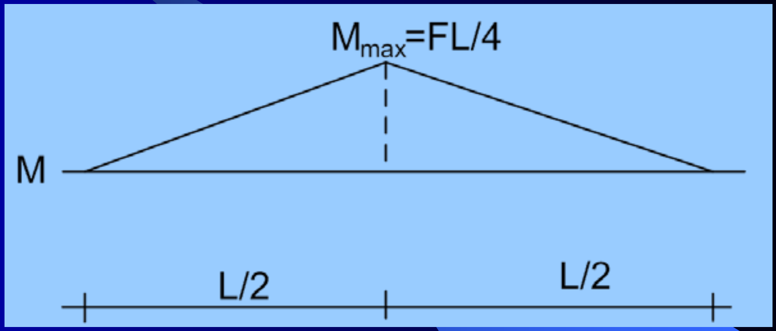
$$\zeta = \sqrt{3\left(1 - \frac{2M_{\max}x}{M_0 L}\right)} = \sqrt{3\left(1 - \frac{Fx}{2M_0}\right)}$$

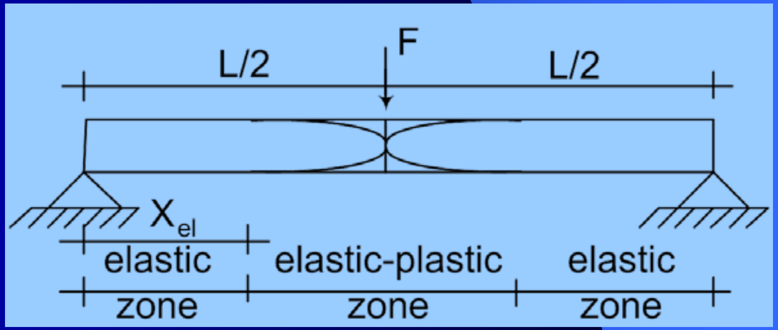
$$[\zeta]_{ult} = \sqrt{3\left(1 - \frac{2x}{L}\right)}$$

Expanded Plastic zone:

Example) Rectangular Cross Section Beam With Elastic Perfectly Plastic Material (continue …):







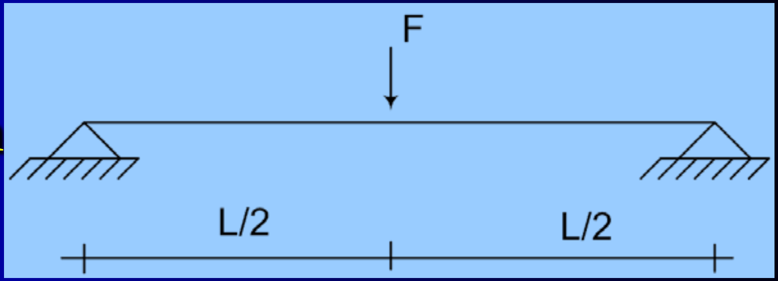
$$x \in \left[0, \frac{4M_0}{3F}\right]: \qquad \kappa = \frac{M}{EI} = \frac{Fx}{2EI}$$

$$\kappa_{el} = \frac{M_{el}}{EI} = \frac{M_0}{\alpha EI} = \frac{2M_0}{3EI}$$

$$x \in \left[\frac{4M_0}{3F}, \frac{L}{2}\right]: \qquad \kappa = \frac{\kappa_{el}}{\sqrt{3\left(1 - \frac{M}{M_0}\right)}}$$

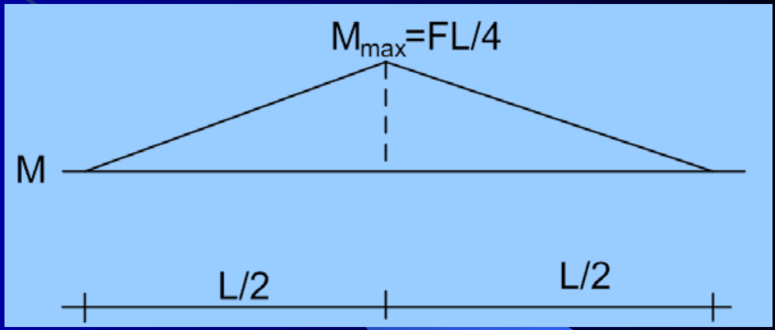
$$\kappa = \frac{2M_0}{3EI\sqrt{3\left(1 - \frac{M}{M_0}\right)}} = \frac{2M_0}{3EI\sqrt{3\left(1 - \frac{Fx}{2M_0}\right)}}$$

Expanded Plastic zone:
Example) Rectangular Cross Section Beam With Elastic Perfectly Plastic Material (continue …):



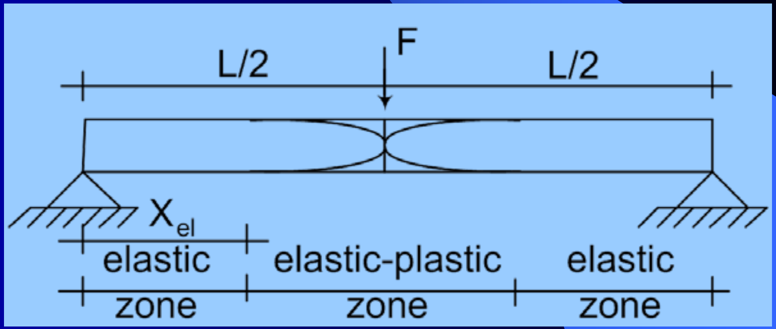
$$x \in \left[0, \frac{4M_0}{3F}\right]:$$

$$\kappa = \frac{M}{EI} = \frac{Fx}{2EI}$$



$$x \in \left[\frac{4M_0}{3F}, \frac{L}{2}\right]:$$

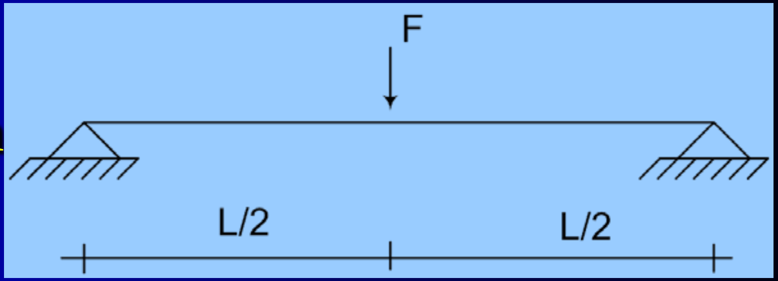
$$\kappa = \frac{2M_0}{3EI\sqrt{3\left(1-\frac{Fx}{2M_0}\right)}}$$



$$F \le F_{el} \rightarrow \Delta_{L/2} = 2\int_0^{\frac{L}{2}} \frac{Fx^2}{4EI}dx$$

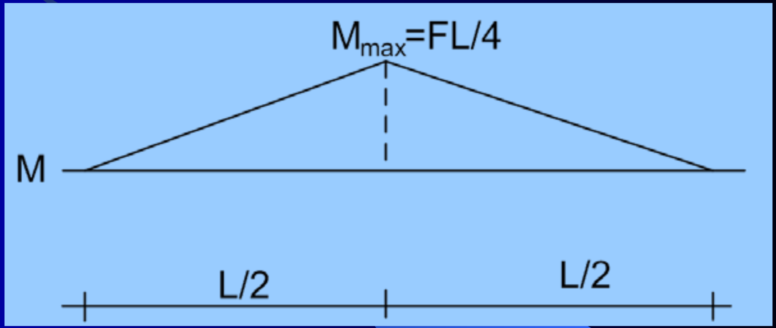
$$F_{el} < F \le F_0 \rightarrow \Delta_{L/2} = 2\int_0^{\frac{4M_0}{3F}} \frac{Fx^2}{4EI}dx + 2\int_{\frac{4M_0}{3F}}^{\frac{L}{2}} \frac{M_0 x}{3EI\sqrt{3\left(1-\frac{Fx}{2M_0}\right)}}dx$$

Expanded Plastic zone:
Example) Rectangular Cross Section Beam With Elastic Perfectly Plastic Material (continue …):

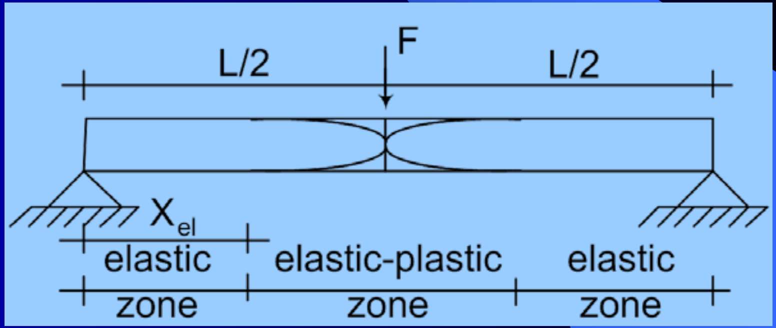


$$F \le F_{el} = \frac{4M_{el}}{L}:$$

$$\Delta_{L/2} = \frac{FL^3}{48EI}$$



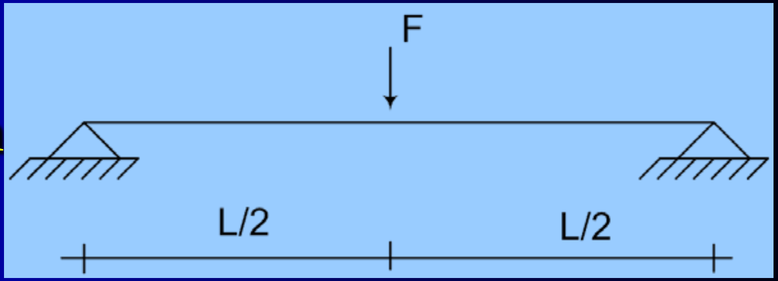
$$F_{el} = \frac{4M_{el}}{L} < F \le F_0 = \frac{4M_0}{L}$$

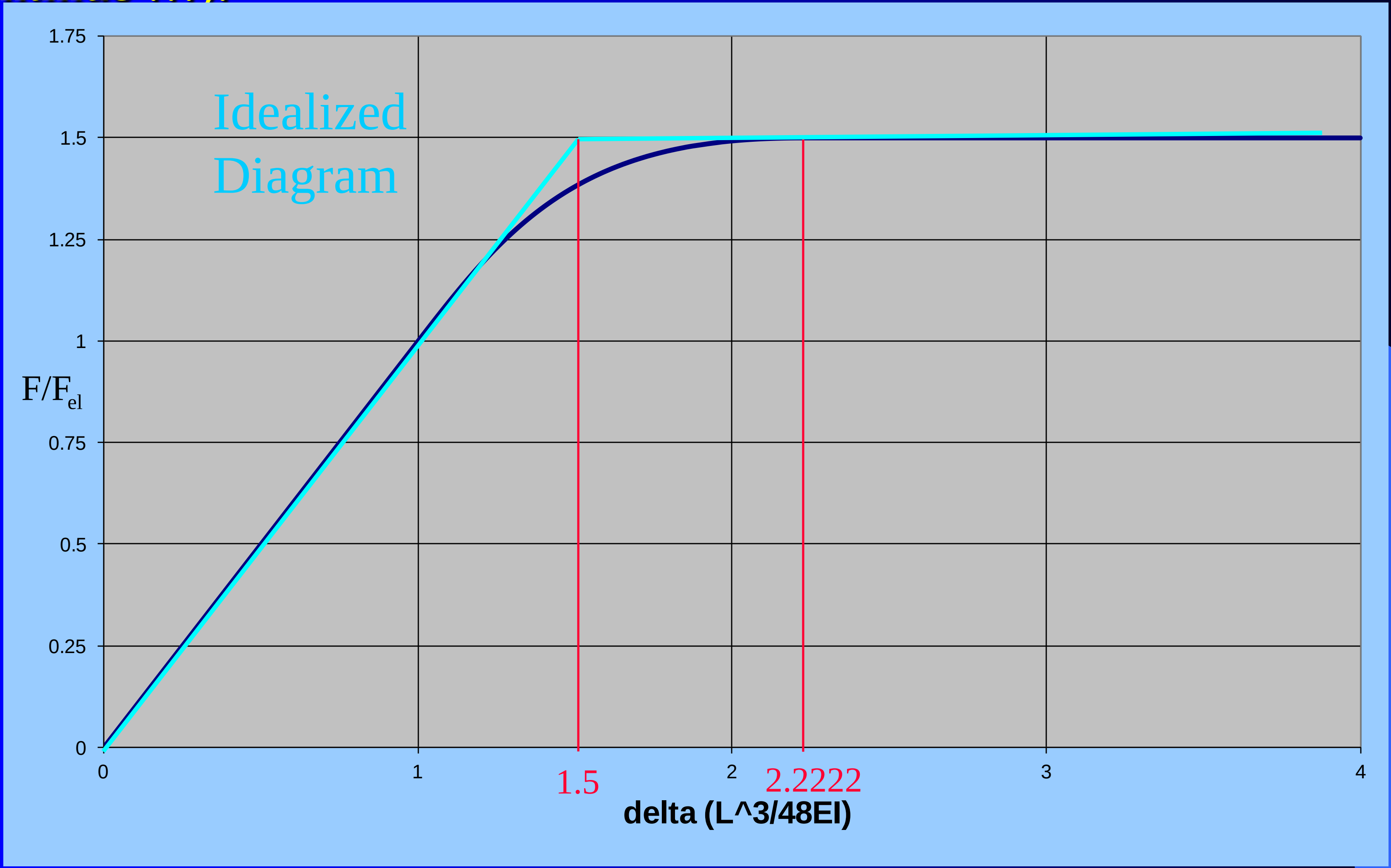


$$\Delta_{L/2} = \frac{2M_0^3}{81F^2EI}\left[80 + \left(12 - \frac{3FL}{M_0}\right)^{\frac{3}{2}} - 36\left(12 - \frac{3FL}{M_0}\right)^{\frac{1}{2}}\right]$$

$$\Delta_{L/2} = \frac{F_0^3L^3}{2592F^2EI}\left[80 + \left(12 - \frac{12F}{F_0}\right)^{\frac{3}{2}} - 36\left(12 - \frac{12F}{F_0}\right)^{\frac{1}{2}}\right]$$

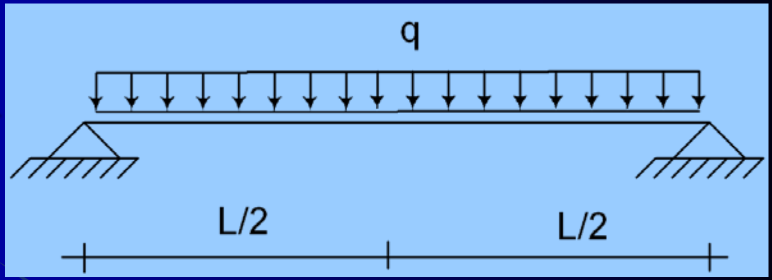
Expanded Plastic zone:
Example) Rectangular Cross Section Beam With Elastic Perfectly Plastic Material (continue …):



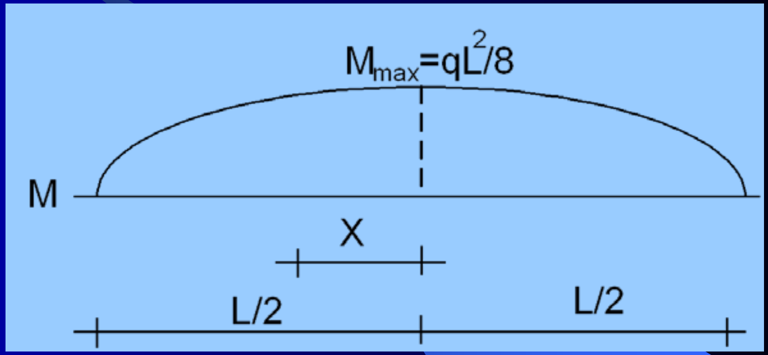


Expanded Plastic zone:

Example) Rectangular Cross Section Beam With Elastic Perfectly Plastic Material:



$$M(x)=\frac{qL^2}{8}-\frac{qx^2}{2},\qquad x\in\left[0,\frac{L}{2}\right]$$

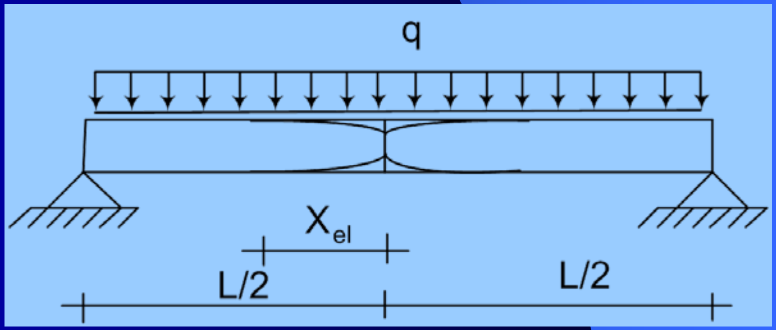


$$x=0\rightarrow M_{\max}=\frac{qL^2}{8}$$

$$q=\frac{8M_{\max}}{L^2}$$

$$q_{el}=\frac{8M_{el}}{L^2}$$

$$q_0=\frac{8M_0}{L^2}$$



$$M(x)=\frac{qL^2}{8}\left(1-4\frac{x^2}{L^2}\right)=M_{\max}\left(1-4\frac{x^2}{L^2}\right),\qquad x\in\left[0,\frac{L}{2}\right]$$

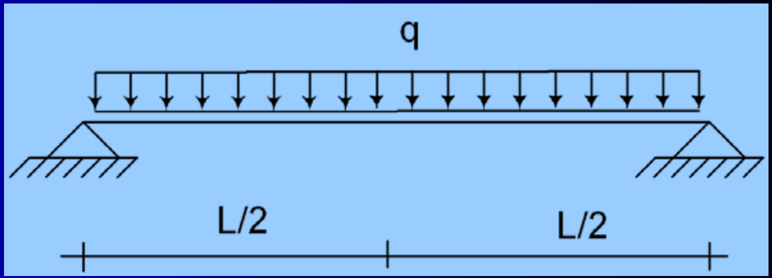
$$x=x_{el}\rightarrow M=M_{el}=\frac{M_0}{\alpha}=\frac{2}{3}M_0$$

$$x_{el}=\sqrt{\frac{L^2}{4}-\frac{4M_0}{3q}}$$

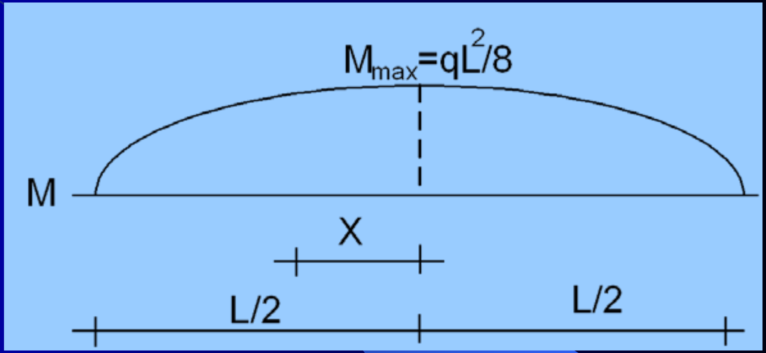
$$\left[x_{el}\right]_{ult}=\frac{L}{2\sqrt{3}}\approx\frac{L}{3.46}$$

Expanded Plastic zone (continue …):

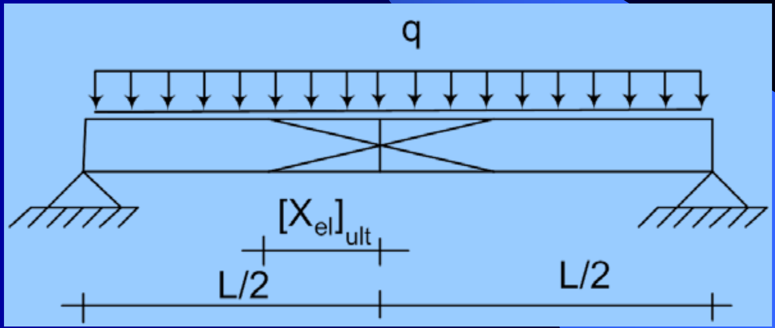
Example) Rectangular Cross Section Beam With Elastic Perfectly Plastic Material:



$$x \in \left[\sqrt{\frac{L^2}{4} - \frac{4M_0}{3q}}, \frac{L}{2} \right] : Elastic\ Region$$



$$x \in \left[0, \sqrt{\frac{L^2}{4} - \frac{4M_0}{3q}} \right] : Elastic - Plastic\ Region$$



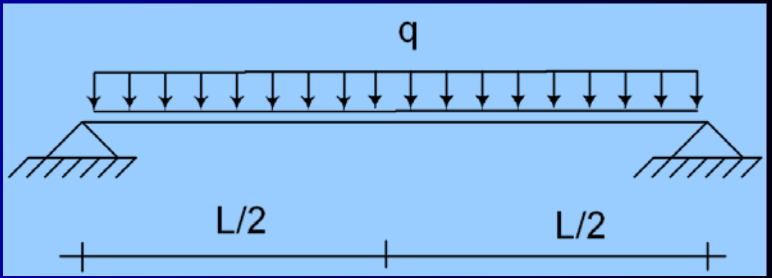
$$M = M_0 \left(1 - \frac{\zeta^2}{3} \right) = M_{\max} \left(1 - 4\frac{x^2}{L^2} \right)$$

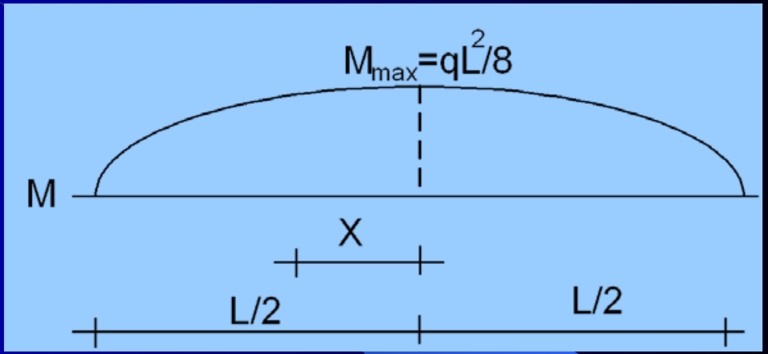
$$\zeta = \pm \sqrt{3 - \frac{3M_{\max}}{M_0} \left(1 - 4\frac{x^2}{L^2} \right)} = \pm \sqrt{3 - \frac{3qL^2}{8M_0} \left(1 - 4\frac{x^2}{L^2} \right)}$$

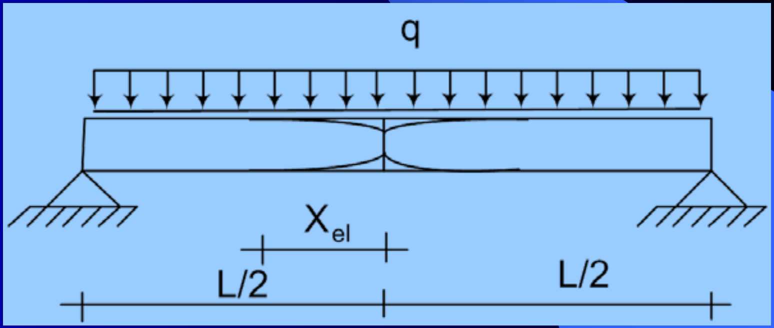
$$[\zeta]_{ult} = \pm 2\sqrt{3}\frac{x}{L}$$

Expanded Plastic zone (continue …):

Example) Rectangular Cross Section Beam With Elastic Perfectly Plastic Material:







$$x \in \left[\sqrt{\frac{L^2}{4} - \frac{4M_0}{3q}}, \frac{L}{2} \right]: \quad \kappa = \frac{M}{EI} = \frac{q}{EI}\left(\frac{L^2}{8} - \frac{x^2}{2}\right)$$

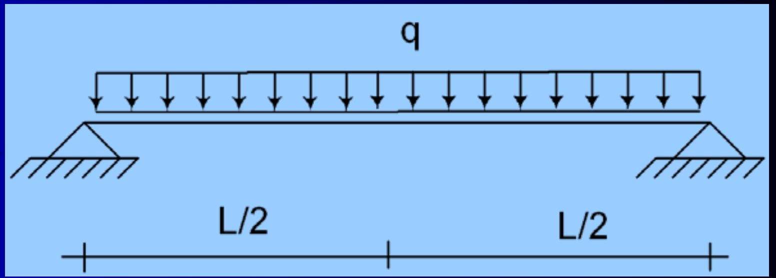
$$\kappa_{el} = \frac{M_{el}}{EI} = \frac{M_0}{\alpha EI} = \frac{2M_0}{3EI}$$

$$x \in \left[0, \sqrt{\frac{L^2}{4} - \frac{4M_0}{3q}} \right]: \quad \kappa = \frac{\kappa_{el}}{\sqrt{3\left(1 - \frac{M}{M_0}\right)}}$$

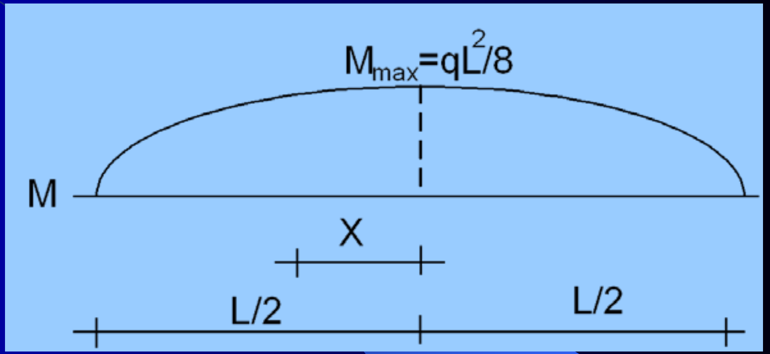
$$\kappa = \frac{2M_0}{3EI\sqrt{3\left(1 - \frac{M}{M_0}\right)}} = \frac{2M_0}{3EI\sqrt{3\left(1 - \frac{qL^2}{8M_0} + \frac{qx^2}{2M_0}\right)}}$$

Expanded Plastic zone (continue …):

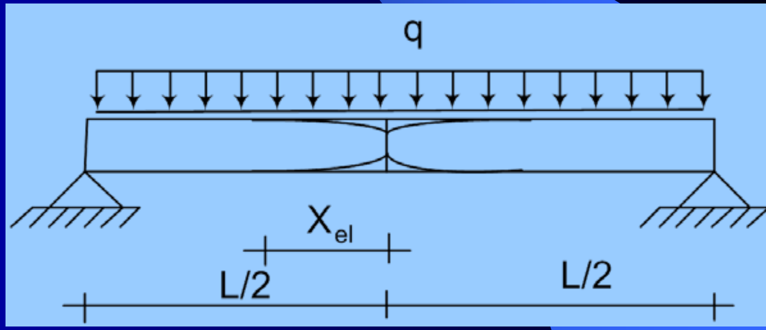
Example) Rectangular Cross Section Beam With Elastic Perfectly Plastic Material:



$$x \in \left[\sqrt{\frac{L^2}{4} - \frac{4M_0}{3q}}, \frac{L}{2} \right] : \kappa = \frac{M}{EI} = \frac{q}{EI}\left(\frac{L^2}{8} - \frac{x^2}{2} \right)$$



$$x \in \left[0, \sqrt{\frac{L^2}{4} - \frac{4M_0}{3q}} \right] : \kappa = \frac{2M_0}{3EI\sqrt{3\left(1 - \frac{qL^2}{8M_0} + \frac{qx^2}{2M_0} \right)}}$$

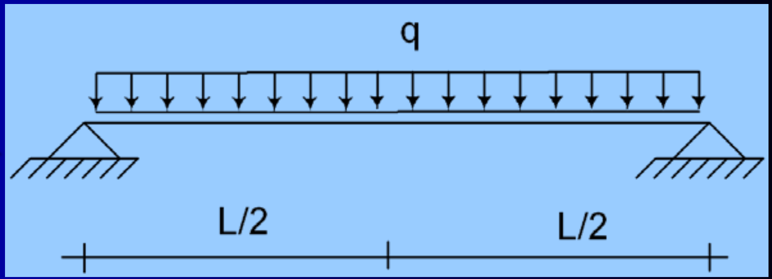


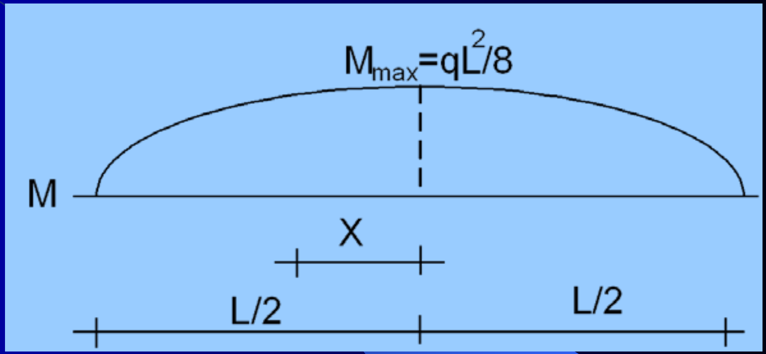
$$q < q_{el} \rightarrow \Delta_{L/2} = 2\int_0^{\frac{L}{2}} \frac{q}{EI}\left(\frac{L^2}{8} - \frac{x^2}{2} \right)\left(\frac{L}{4} - \frac{x}{2} \right) dx$$

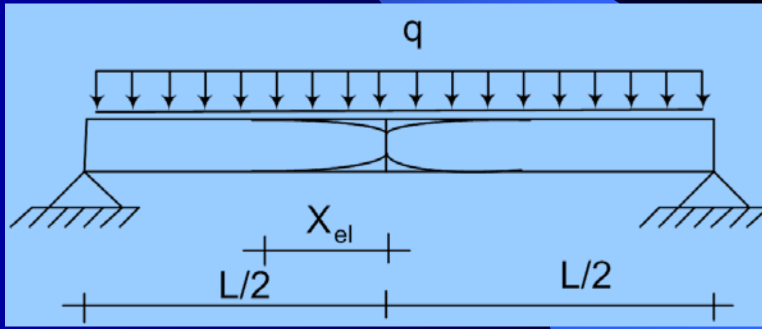
$$q_{el} < q \le q_0 \rightarrow \Delta_{L/2} = 2\int_0^{\sqrt{\frac{L^2}{4} - \frac{4M_0}{3q}}} \frac{2M_0\left(\frac{L}{4} - \frac{x}{2} \right)}{3EI\sqrt{3\left(1 - \frac{qL^2}{8M_0} + \frac{qx^2}{2M_0} \right)}} dx + 2\int_{\sqrt{\frac{L^2}{4} - \frac{4M_0}{3q}}}^{\frac{L}{2}} \frac{q}{EI}\left(\frac{L^2}{8} - \frac{x^2}{2} \right)\left(\frac{L}{4} - \frac{x}{2} \right) dx$$

Expanded Plastic zone (continue …):

Example) Rectangular Cross Section Beam With Elastic Perfectly Plastic Material:







$$q < q_{el} = \frac{8M_{el}}{L^2}:$$

$$\Delta_{L/2} = \frac{5qL^4}{384EI}$$

$$q_{el} = \frac{8M_{el}}{L^2} < q \leq q_0 = \frac{8M_0}{L^2}$$

$$\Delta_{L/2} = ??$$

Dissipation Power in Plastic Hinges:

$$e_p = z\theta$$

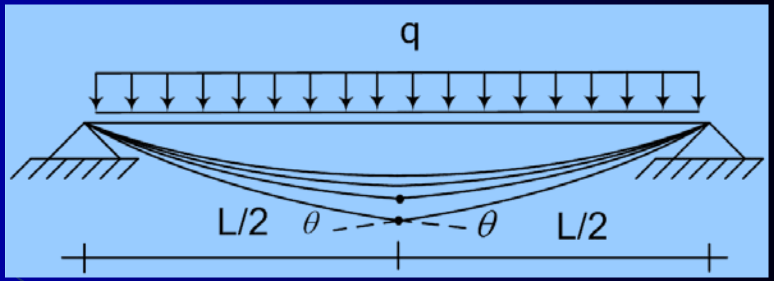
$$D = \sigma\dot{\varepsilon}_p$$

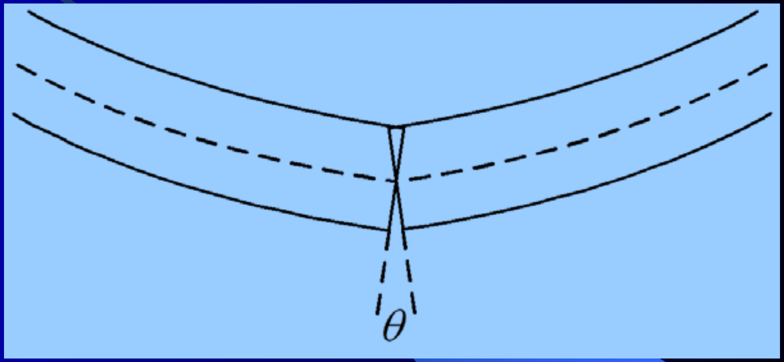
$$D_{\text{int}} = \int_V D\,dV$$

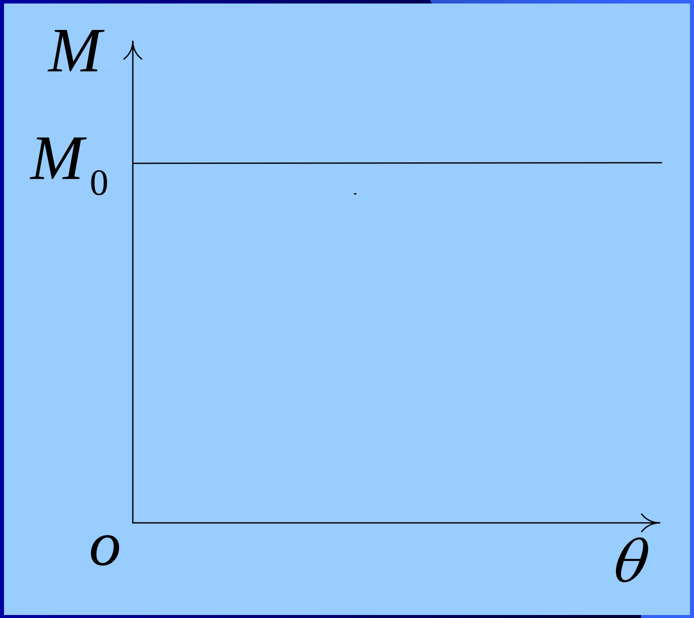
$$D_{\text{int}} = \int_V \sigma\dot{\varepsilon}_p\,dV = \int_A \sigma\dot{e}_p\,dA$$

$$D_{\text{int}} = \int_A \sigma z\dot{\theta}\,dA = \dot{\theta}\int_A \sigma z\,dA = \dot{\theta}M$$

$$D_{\text{int}} = M\dot{\theta} = M_0\left|\dot{\theta}\right|$$







Common Representation of Dissipation Power:

- The Stress-Stain Element: $$D = \sigma \dot{\varepsilon}_p = \sigma_0 \left| \dot{\varepsilon}_p \right|$$

- The Plastic Axial Bar: $$D_{\text{int}} = S \dot{e}_p = S_0 \left| \dot{e}_p \right|$$

- The Plastic Moment Hinge: $$D_{\text{int}} = M \dot{\theta} = M_0 \left| \dot{\theta} \right|$$

Design Methods:

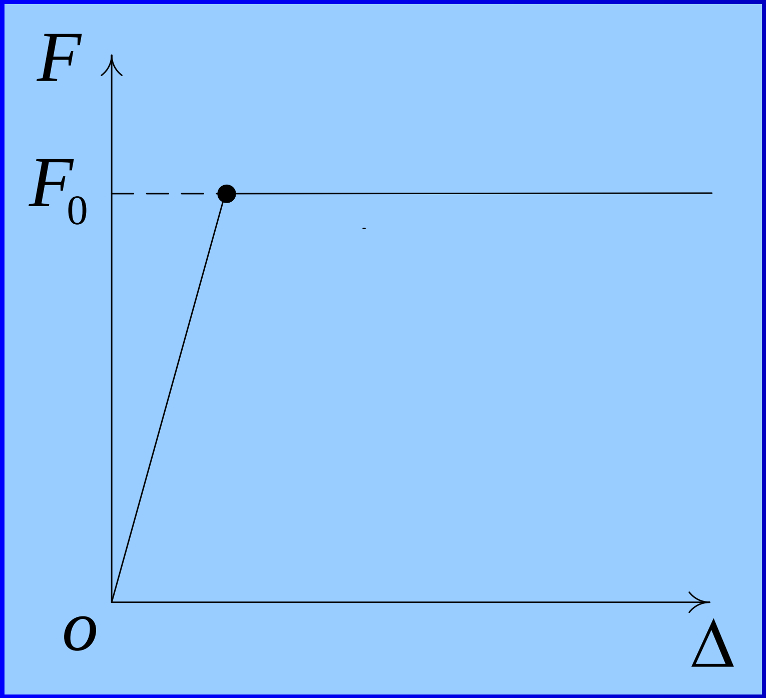
1- Allowable Stress Design or Work Stress Design

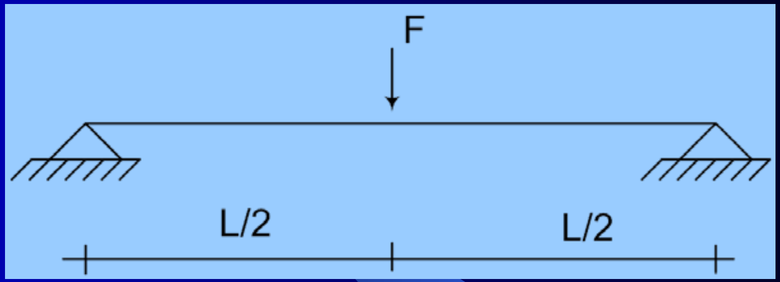
Elastic Analysis is permissible only.

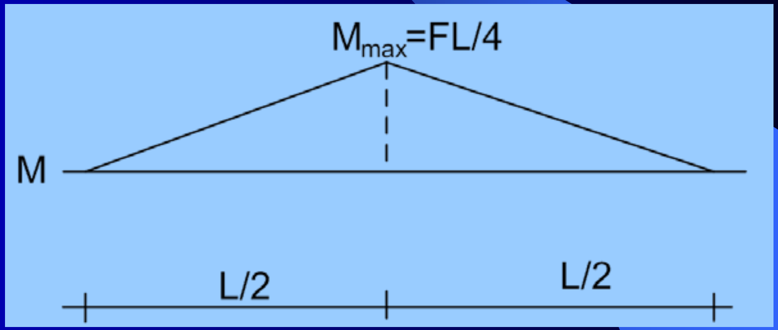
2- Limit States Design or Strength Design

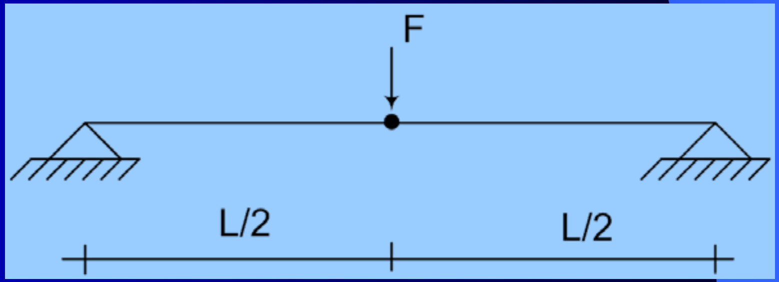
Both of Elastic and Inelastic Analyses are permissible.

Comparison Between Elastic and Inelastic Analysis in Strength Design:

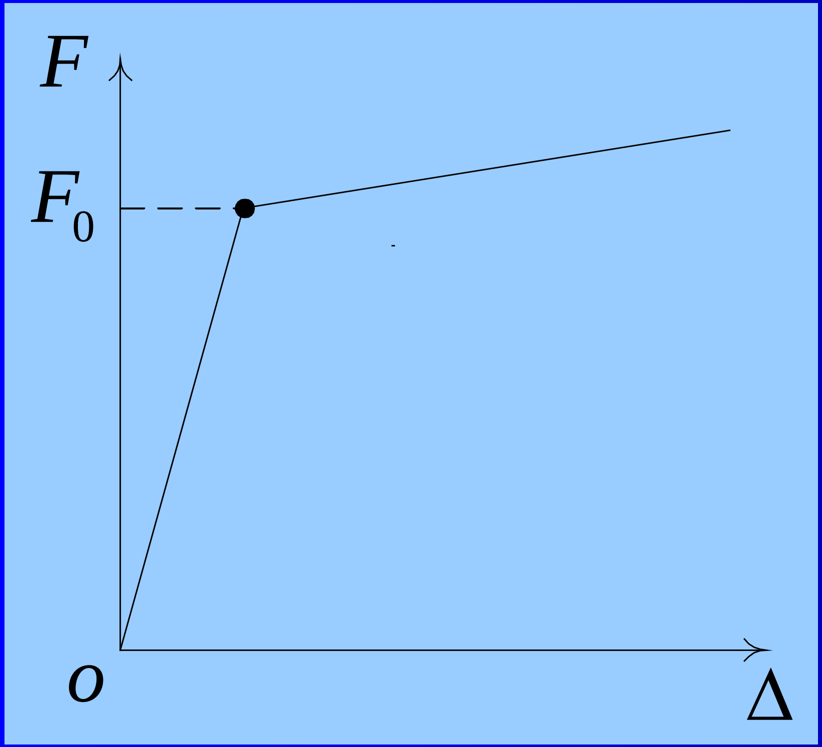


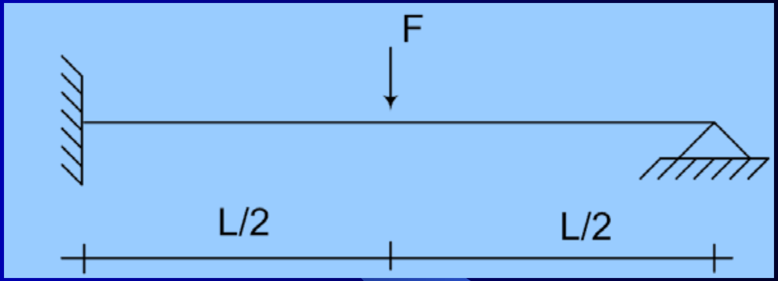


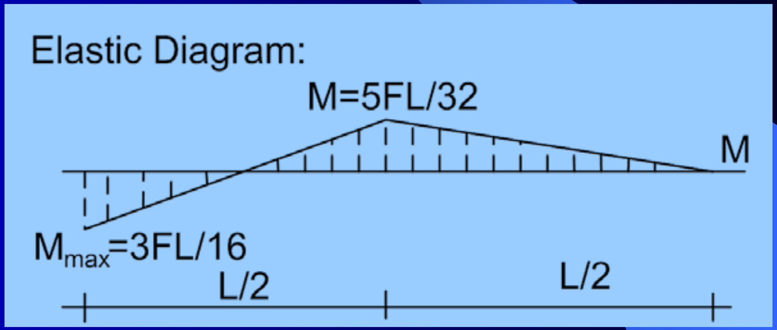


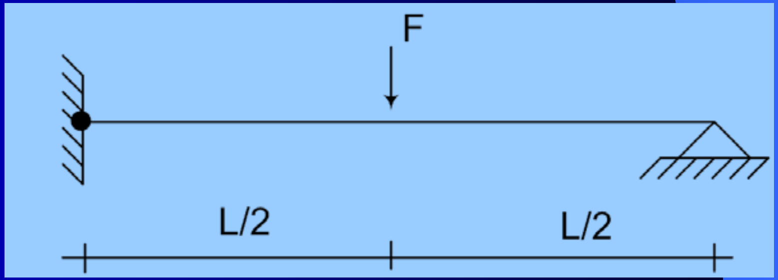


Comparison Between Elastic and Inelastic Analysis in Strength Design:

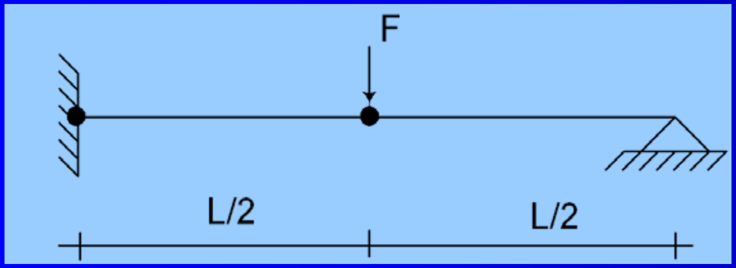


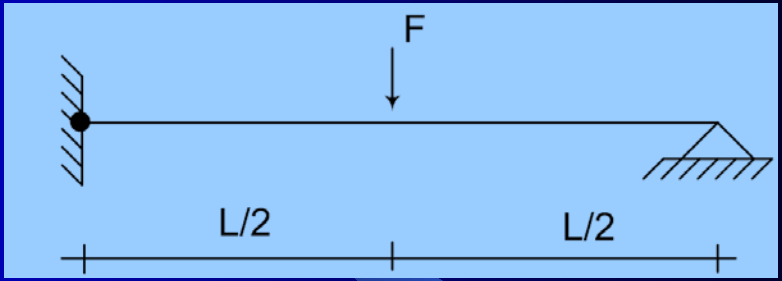


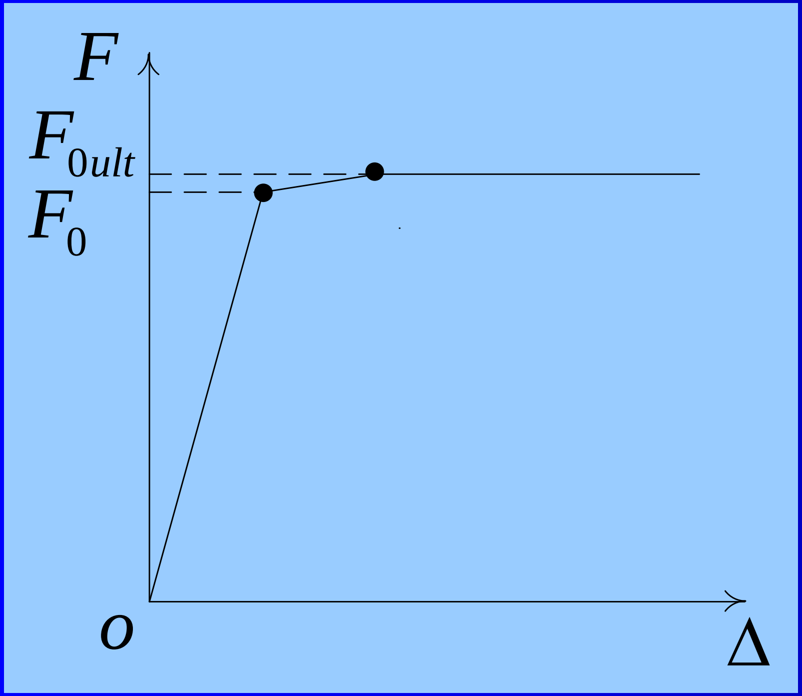


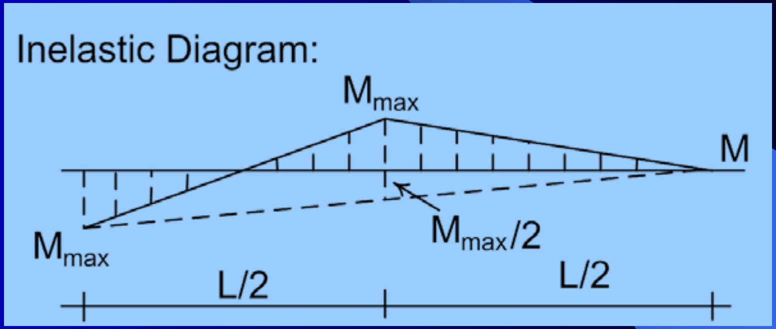


Comparison Between Elastic and Inelastic Analysis (continue …):





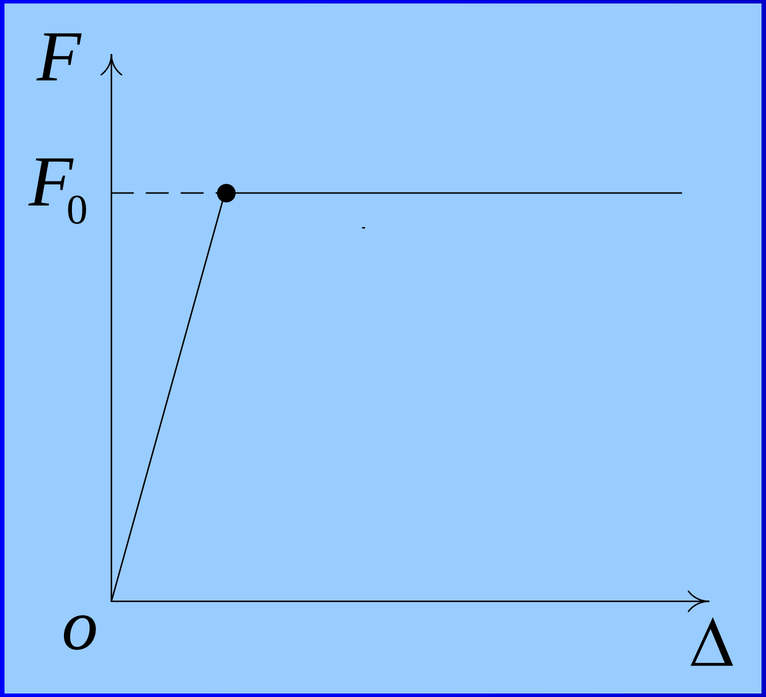


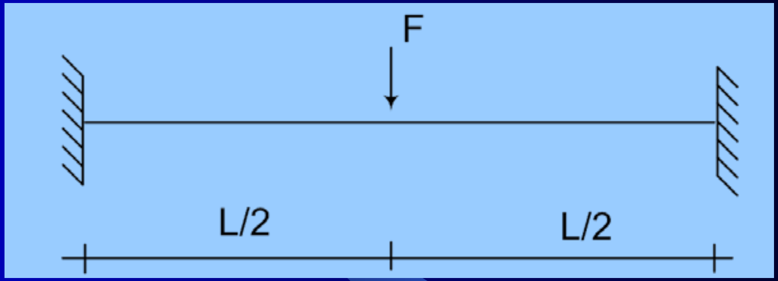


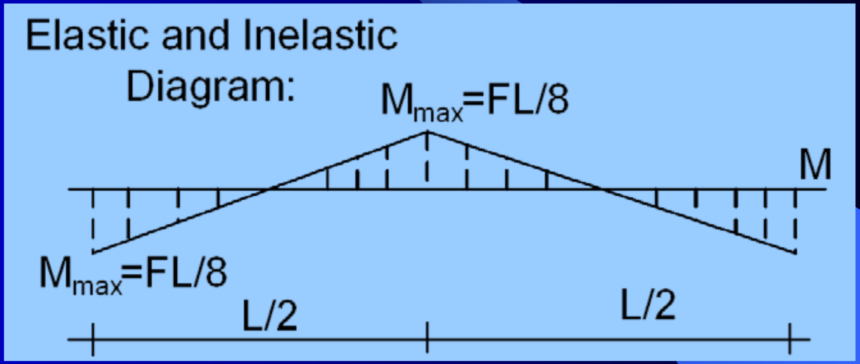
$$\frac{M_{\max}}{2} + M_{\max} = \frac{FL}{4}$$

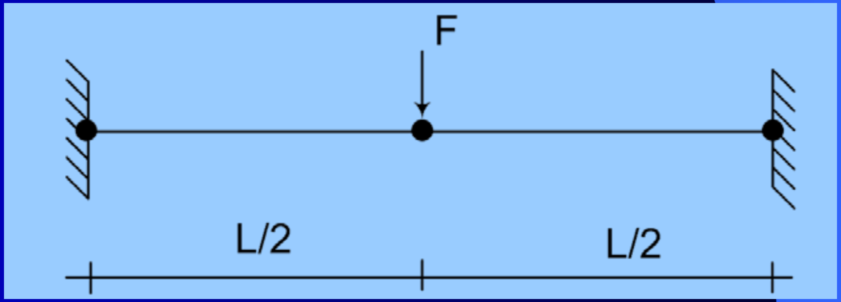
$$M_{\max} = \frac{FL}{6}$$

Comparison Between Elastic and Inelastic Analysis (continue …):







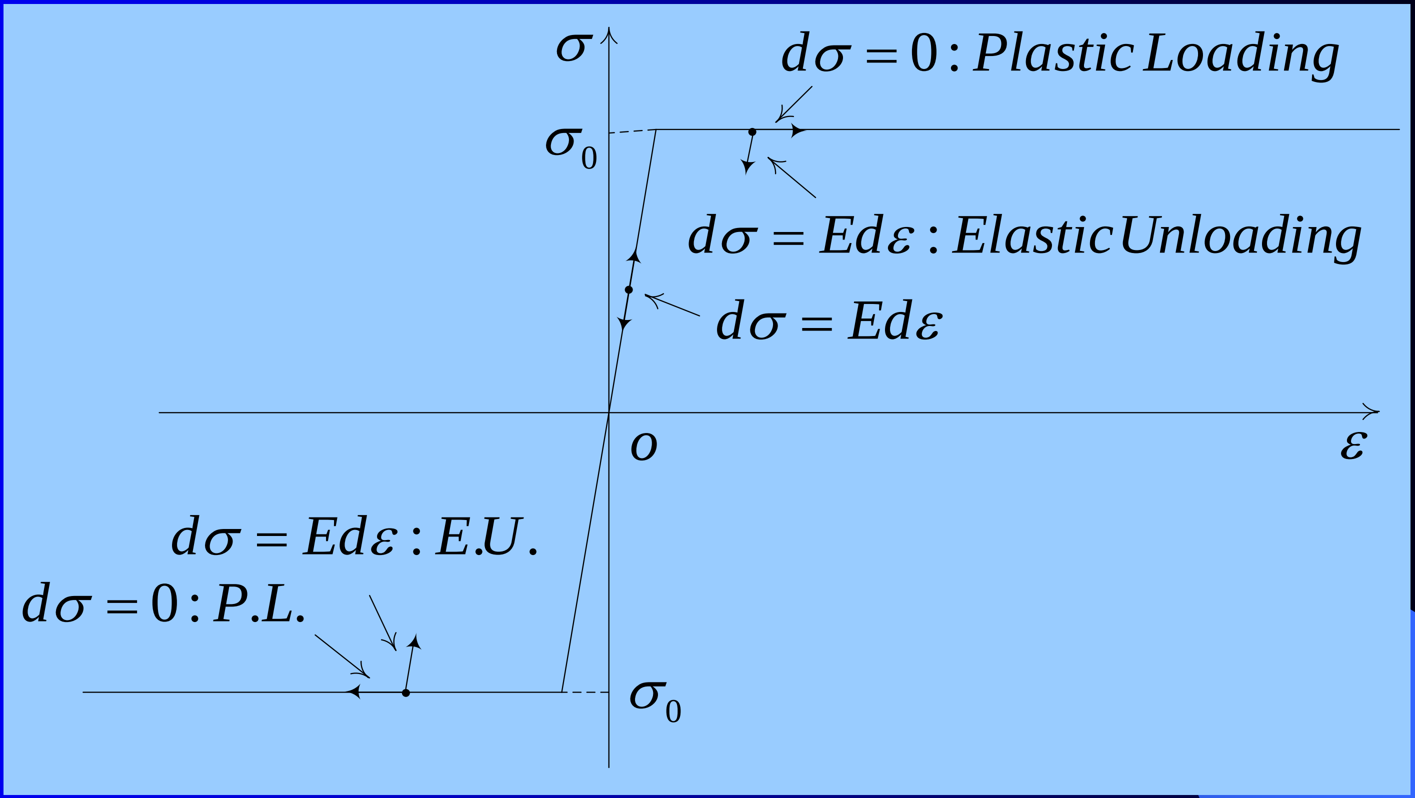


Two Types of Nonlinear Analyses used in Lumped Hinge Approach:

1- Incremental Analysis

2- Limit Analysis

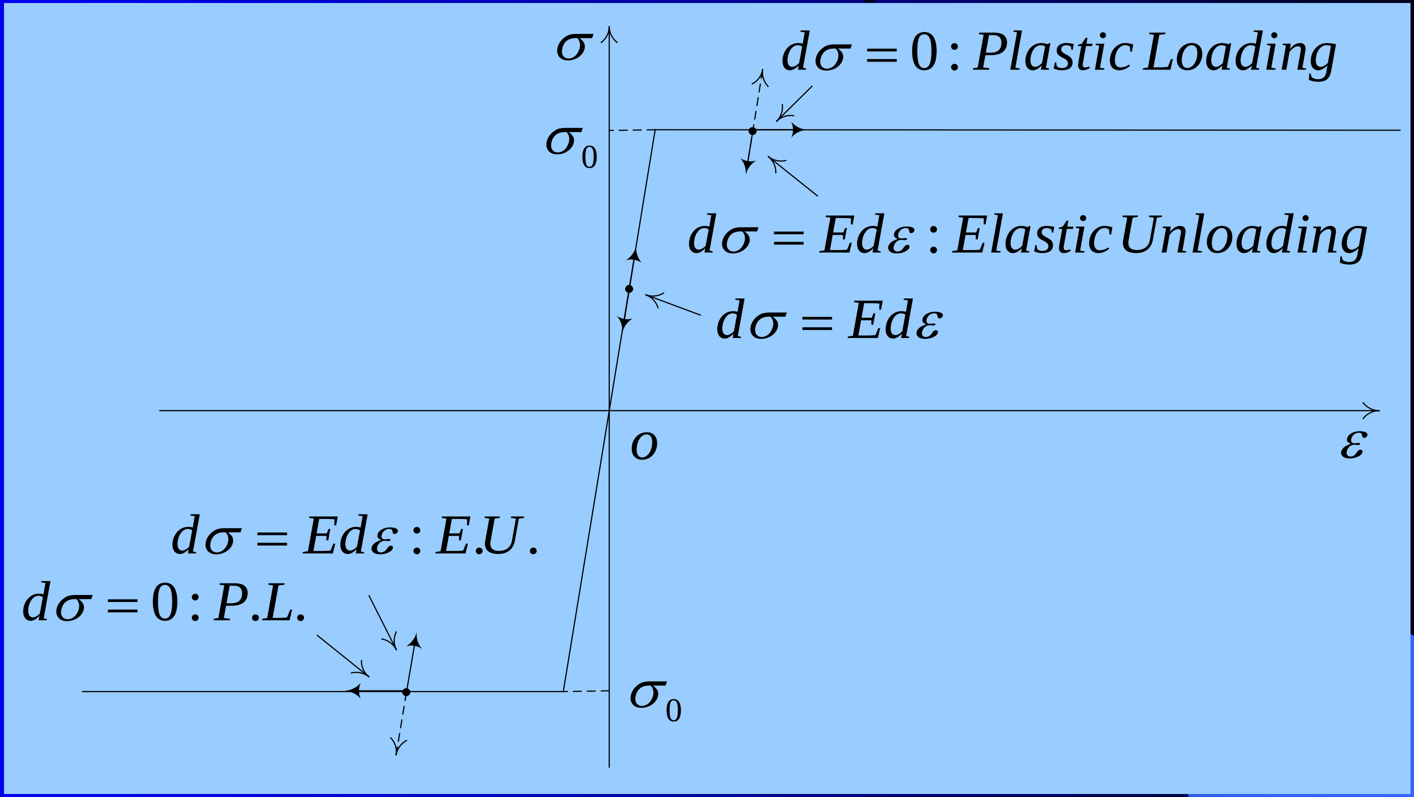
Incremental Analysis:



Elastic Unloading – Plastic Loading Criteria?

Incremental Analysis:

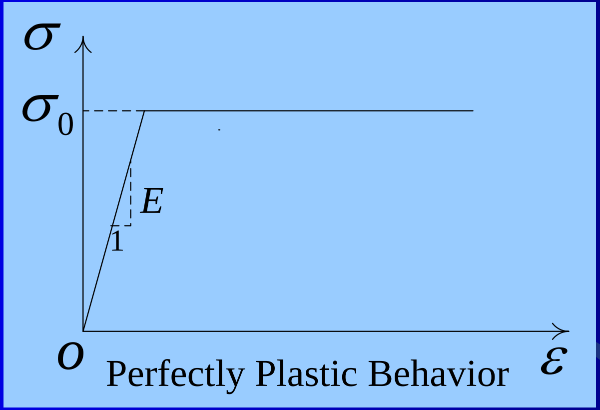
Elastic Predictor – Plastic Corrector for Elastic Perfectly Plastic Behavior

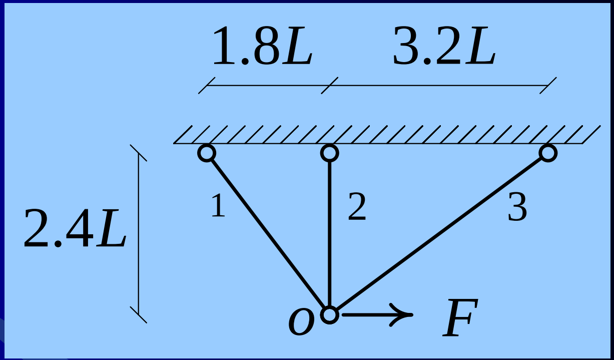


$$Elastic\ Region :$$
$$d\sigma = Ed\varepsilon$$

$$Yield\ Region :$$
$$d\sigma = Ed\varepsilon\left(Elastic\ Predictor\right) \rightarrow \begin{cases} if\ \sigma d\sigma < 0 : Elastic\ Unloading \\ else : Plastic\ Loading\left(Plastic\ Corrector\right) \end{cases}$$

Incremental Analysis, Example:





$$\begin{cases} A_1 = 0.90A \\ A_2 = 0.96A \\ A_3 = A \end{cases}$$

$$\begin{cases} \sigma_{y1} = 0.8\sigma_0 \\ \sigma_{y2} = 0.06\sigma_0 \\ \sigma_{y3} = \sigma_0 \end{cases}$$

$$\begin{cases} \vec{U}_{xo} = u \\ \downarrow U_{yo} = v \end{cases}$$

$$\frac{EA}{L} = K$$

$$\sigma_0 A = S_0$$

$$Kinematic\ Equations: \begin{cases} \dot{e}_1 = 0.6\dot{u} + 0.8\dot{v} \\ \dot{e}_2 = \dot{v} \\ \dot{e}_3 = -0.8\dot{u} + 0.6\dot{v} \end{cases}$$

$$\dot{\mathbf{e}} = \mathbf{B}\dot{\mathbf{d}}$$

$$\begin{matrix} Kinematic \\ Matrix \end{matrix} : \mathbf{B} = \begin{bmatrix} 0.6 & 0.8 \\ 0 & 1 \\ -0.8 & 0.6 \end{bmatrix}$$

$$Equilibrium\ Equations: \begin{cases} \sum \dot{F}_x = \dot{F} \\ \sum \dot{F}_y = 0 \end{cases} \rightarrow \begin{cases} 0.6\dot{S}_1 - 0.8\dot{S}_3 = \dot{F} \\ 0.8\dot{S}_1 + \dot{S}_2 + 0.6\dot{S}_3 = 0 \end{cases}$$

$$\mathbf{B}^T \dot{\mathbf{s}} = \dot{\mathbf{f}}$$

$$\mathbf{B}^T : \begin{matrix} Static \\ Matrix \end{matrix}$$

Step 1:

$$Constitutive\ Equations:\begin{cases}\dot{S}_1 = 0.90EA\dfrac{\dot{e}_1}{L_1} = 0.90EA\dfrac{\dot{e}_1}{3L} = 0.30K\dot{e}_1\\ \dot{S}_2 = 0.96EA\dfrac{\dot{e}_2}{L_2} = 0.96EA\dfrac{\dot{e}_2}{2.4L} = 0.40K\dot{e}_2\\ \dot{S}_3 = EA\dfrac{\dot{e}_3}{L_3} = EA\dfrac{\dot{e}_3}{4L} = 0.25K\dot{e}_3\end{cases}$$

$$\dot{\mathbf{s}} = \mathbf{D}^{(1)}\dot{\mathbf{e}}$$

$$Initial\ Generalized\ Material\ Stiffness\ Matrix:\mathbf{D}^{(1)} = \begin{bmatrix}\dfrac{0.9EA}{L_1} & 0 & 0\\ 0 & \dfrac{0.96EA}{L_2} & 0\\ 0 & 0 & \dfrac{EA}{L_3}\end{bmatrix}$$

Step 1 (continue…):

$$\begin{cases} \mathbf{B}^T \dot{\mathbf{s}} = \dot{\mathbf{f}} \\ \dot{\mathbf{s}} = \mathbf{D}^{(1)} \dot{\mathbf{e}} \end{cases}$$

$$\mathbf{B}^T \mathbf{D}^{(1)} \dot{\mathbf{e}} = \dot{\mathbf{f}}$$

$$\begin{cases} 0.18\dot{e}_1 - 0.20\dot{e}_3 = \dot{F}\dfrac{L}{EA} \\ 0.24\dot{e}_1 + 0.40\dot{e}_2 + 0.15\dot{e}_3 = 0 \end{cases}$$

$$\begin{cases} \mathbf{B}^T \mathbf{D}^{(1)} \dot{\mathbf{e}} = \dot{\mathbf{f}} \\ \dot{\mathbf{e}} = \mathbf{B}\dot{\mathbf{d}} \end{cases}$$

$$\mathbf{B}^T \mathbf{D}^{(1)} \mathbf{B}\dot{\mathbf{d}} = \mathbf{K}^{(1)} \dot{\mathbf{d}} = \dot{\mathbf{f}}$$

$$\begin{cases} 0.268\dot{u} + 0.024\dot{v} = \dfrac{\dot{F}}{K} \\ 0.024\dot{u} + 0.682\dot{v} = 0 \end{cases}$$

$$\begin{cases} \dot{u} = +3.743\,\dot{F}/K \\ \dot{v} = -0.132\,\dot{F}/K \end{cases}$$

$$\begin{cases} \dot{\mathbf{s}} = \mathbf{D}^{(1)} \dot{\mathbf{e}} \\ \dot{\mathbf{e}} = \mathbf{B}\dot{\mathbf{d}} \end{cases}$$

$$\dot{\mathbf{s}} = \mathbf{D}^{(1)} \mathbf{B}\dot{\mathbf{d}}$$

$$\begin{cases} \dot{S}_1 = +0.642\dot{F} \\ \dot{S}_2 = -0.053\dot{F} \\ \dot{S}_3 = -0.768\dot{F} \end{cases}$$

$$\begin{cases} \left|S_1/S_{y1}\right| = \left(0 + 0.642\dot{F}\right)/\left(0.9A \times 0.8\sigma_0\right) = 0.892\,\dot{F}/S_0 \\ \left|S_2/S_{y2}\right| = \left(0 + 0.053\dot{F}\right)/\left(0.96A \times 0.06\sigma_0\right) = 0.920\,\dot{F}/S_0 \\ \left|S_3/S_{y3}\right| = \left(0 + 0.768\dot{F}\right)/\left(A \times \sigma_0\right) = 0.768\,\dot{F}/S_0 \end{cases}$$

$: Max$

$$0.920\,\dot{F}_{(1)}/S_0 = 1$$

Step 1 (continue…):

$$\dot{F}_{(1)} = (1/0.92)S_0 = 1.087S_0$$

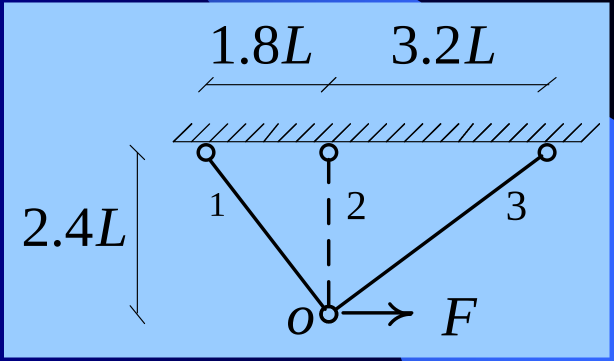
$$F_{(1)} = 0 + 1.087S_0 = 1.087S_0$$

$$\begin{cases} S_{1(1)} = 0 + 0.642\dot{F}_{(1)} = 0 + 0.698S_0 \\ S_{2(1)} = 0 - 0.053\dot{F}_{(1)} = 0 - 0.058S_0 \\ S_{3(1)} = 0 - 0.768\dot{F}_{(1)} = 0 - 0.835S_0 \end{cases}$$

$$\begin{cases} u_{(1)} = 0 + 3.743\dfrac{\dot{F}_{(1)}L}{EA} = 0 + 4.069\dfrac{S_0}{K} \\ v_{(1)} = 0 - 0.132\dfrac{\dot{F}_{(1)}L}{EA} = 0 - 0.143\dfrac{S_0}{K} \end{cases}$$

Step 2:

$$Constitutive\ Equations: \begin{cases} \dot{S}_1 = 0.30K\dot{e}_1 : (Elastic) \\ \dot{S}_2 = 0 : (Plastic) \\ \dot{S}_3 = 0.25K\dot{e}_3 : (Elastic) \end{cases}$$



$$\dot{\mathbf{s}} = \mathbf{D}^{(2)}\dot{\mathbf{e}}$$

$$\mathbf{B}^T\mathbf{D}^{(2)}\dot{\mathbf{e}} = \dot{\mathbf{f}}$$

$$\begin{cases} 0.18\dot{e}_1 - 0.20\dot{e}_3 = \dot{F}/K \\ 0.24\dot{e}_1 + 0.15\dot{e}_3 = 0 \end{cases}$$

Step 2 (continue…):

$$\mathbf{B}^T\mathbf{D}^{(2)}\mathbf{B}\dot{\mathbf{d}} = \mathbf{K}^{(2)}\dot{\mathbf{d}} = \dot{\mathbf{f}}$$

$$\begin{cases} 0.268\dot{u} + 0.024\dot{v} = \dot{F}/K \\ 0.024\dot{u} + 0.282\dot{v} = 0 \end{cases}$$

$$\begin{cases} \dot{u} = +3.760\,\dot{F}/K \\ \dot{v} = -0.320\,\dot{F}/K \end{cases}$$

$$\dot{\mathbf{s}} = \mathbf{D}^{(2)}\mathbf{B}\dot{\mathbf{d}}$$

$$\begin{cases} \dot{S}_1 = +0.600\dot{F} \\ \dot{S}_2 = 0 \\ \dot{S}_3 = -0.800\dot{F} \end{cases}$$

$$\begin{cases} \left|S_1/S_{y1}\right| = \left|0.698S_0 + 0.6\dot{F}\right|/\left(0.9A\times 0.8\sigma_0\right) = 0.969 + 0.833\,\dot{F}/S_0 \\ \left|S_2/S_{y2}\right| = 1 \\ \left|S_3/S_{y3}\right| = \left|-0.835S_0 - 0.8\dot{F}\right|/\left(A\times\sigma_0\right) = 0.835 + 0.8\,\dot{F}/S_0 \end{cases}$$

$$: Max$$

$$0.969 + 0.833\,\dot{F}_{(2)}/S_0 = 1$$

$$\dot{F}_{(2)} = \left(\frac{1-0.969}{0.833}\right)S_0 = 0.037S_0$$

$$F_{(2)} = 1.087S_0 + 0.037S_0 = 1.124S_0$$

Step 2 (continue…):

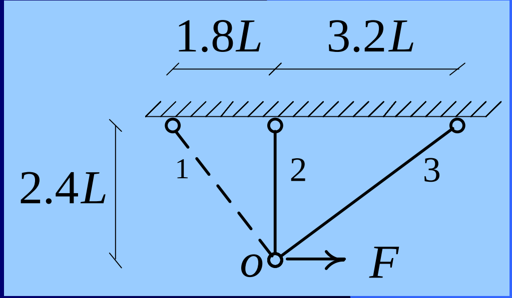
$$\begin{cases} S_{1(2)} = +0.698S_0 + 0.022S_0 = 0.720S_0 \\ S_{2(2)} = -0.058S_0 \\ S_{3(2)} = -0.835S_0 - 0.030S_0 = -0.865S_0 \end{cases}$$

$$\begin{cases} \dot{u}_{(2)} = +3.760\left(0.037S_0\right)/K = +0.139\,S_0/K \\ \dot{v}_{(2)} = -0.320\left(0.037S_0\right)/K = -0.012\,S_0/K \end{cases}$$

$$\begin{cases} u_{(2)} = +4.069\dfrac{S_0}{K} + 0.139\dfrac{S_0}{K} = +4.208\dfrac{S_0}{K} \\ v_{(2)} = -0.143\dfrac{S_0}{K} - 0.012\dfrac{S_0}{K} = -0.155\dfrac{S_0}{K} \end{cases}$$

Step 3:

$$Constitutive\ Equations: \begin{cases} \dot{S}_1 = 0 : (Plastic) \\ \dot{S}_2 = 0.40K\dot{e}_2 : (Elastic\ Predictor) \\ \dot{S}_3 = 0.25K\dot{e}_3 : (Elastic) \end{cases}$$



$$\dot{\mathbf{s}} = \mathbf{D}_{ep}^{(3)}\dot{\mathbf{e}}$$

Step 3 (continue…):

$$\mathbf{B}^T\mathbf{D}_{ep}^{(3)}\dot{\mathbf{e}}=\dot{\mathbf{f}}$$

$$\begin{cases}-0.2\dot{e}_3=\dot{F}/K\\ 0.4\dot{e}_2+0.15\dot{e}_3=0\end{cases}$$

$$\mathbf{B}^T\mathbf{D}_{ep}^{(3)}\mathbf{B}\dot{\mathbf{d}}=\mathbf{K}_{ep}^{(3)}\dot{\mathbf{d}}=\dot{\mathbf{f}}$$

$$\begin{cases}0.16\dot{u}-0.12\dot{v}=\dot{F}/K\\ -0.12\dot{u}+0.49\dot{v}=0\end{cases}$$

$$\begin{cases}\dot{u}=+7.656\,\dot{F}/K\\ \dot{v}=+1.875\,\dot{F}/K\end{cases}$$

$$\dot{\mathbf{s}}=\mathbf{D}_{ep}^{(3)}\mathbf{B}\dot{\mathbf{d}}$$

$$\begin{cases}\dot{S}_1=0\\ \dot{S}_2=+0.75\dot{F}:(S_2\dot{S}_2<0\rightarrow Elastic\,Unloading)\\ \dot{S}_3=-1.25\dot{F}\end{cases}$$

$$\mathbf{D}^{(3)}=\mathbf{D}_{ep}^{(3)}$$

$$\begin{cases}\left|S_1/S_{y1}\right|=1\\ \left|S_2/S_{y2}\right|=\left|-0.058S_0+0.75\dot{F}\right|/(0.96A\times 0.06\sigma_0)=-1+13.02\,\dot{F}/S_0\\ \left|S_3/S_{y3}\right|=\left|-0.865S_0-1.25\dot{F}\right|/(A\times\sigma_0)=0.865+1.25\,\dot{F}/S_0\end{cases}$$

$$-1+13.02\,\dot{F}/S_0=1 \qquad \dot{F}=0.154S_0$$

$$0.865+1.25\,\dot{F}/S_0=1 \qquad \dot{F}=0.108S_0 \qquad :Min$$

Step 3 (continue…):

$$\dot{F}_{(3)} = 0.108 S_0$$

$$F_{(3)} = 1.124 S_0 + 0.108 S_0 = 1.232 S_0$$

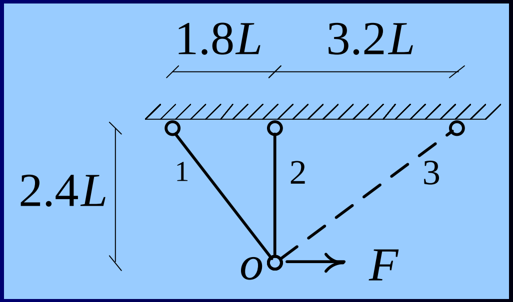
$$\begin{cases} S_{1(3)} = +0.720 S_0 \\ S_{2(3)} = -0.058 S_0 + 0.081 S_0 = +0.023 S_0 \\ S_{3(3)} = -0.865 S_0 - 0.135 S_0 = -S_0 \end{cases}$$

$$\begin{cases} \dot{u}_{(3)} = +7.656\left(0.108 S_0\right)/K = +0.827\, S_0/K \\ \dot{v}_{(3)} = +1.875\left(0.108 S_0\right)/K = +0.203\, S_0/K \end{cases}$$

$$\begin{cases} u_{(3)} = +4.208 \dfrac{S_0}{K} + 0.827 \dfrac{S_0}{K} = +5.035 \dfrac{S_0}{K} \\ v_{(3)} = -0.155 \dfrac{S_0}{K} + 0.203 \dfrac{S_0}{K} = +0.048 \dfrac{S_0}{K} \end{cases}$$

Step 4:

$$\text{Constitutive Equations}:\begin{cases}\dot{S}_1 = 0.30K\dot{e}_1 : (\text{Elastic Predictor}) \\ \dot{S}_2 = 0.40K\dot{e}_2 : (\text{Elastic}) \\ \dot{S}_3 = 0 : (\text{Plastic})\end{cases}$$



$$\dot{\mathbf{s}} = \mathbf{D}_{ep}^{(4)}\dot{\mathbf{e}}$$

$$\mathbf{B}^T\mathbf{D}_{ep}^{(4)}\dot{\mathbf{e}} = \dot{\mathbf{f}}$$

$$\begin{cases}0.18\dot{e}_1 = \dot{F}/K \\ 0.24\dot{e}_1 + 0.4\dot{e}_2 = 0\end{cases}$$

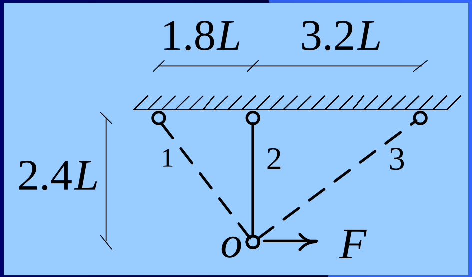
$$\mathbf{B}^T\mathbf{D}_{ep}^{(4)}\mathbf{B}\dot{\mathbf{d}} = \mathbf{K}_{ep}^{(4)}\dot{\mathbf{d}} = \dot{\mathbf{f}}$$

$$\begin{cases}0.108\dot{u} + 0.144\dot{v} = \dot{F}/K \\ 0.144\dot{u} + 0.592\dot{v} = 0\end{cases}$$

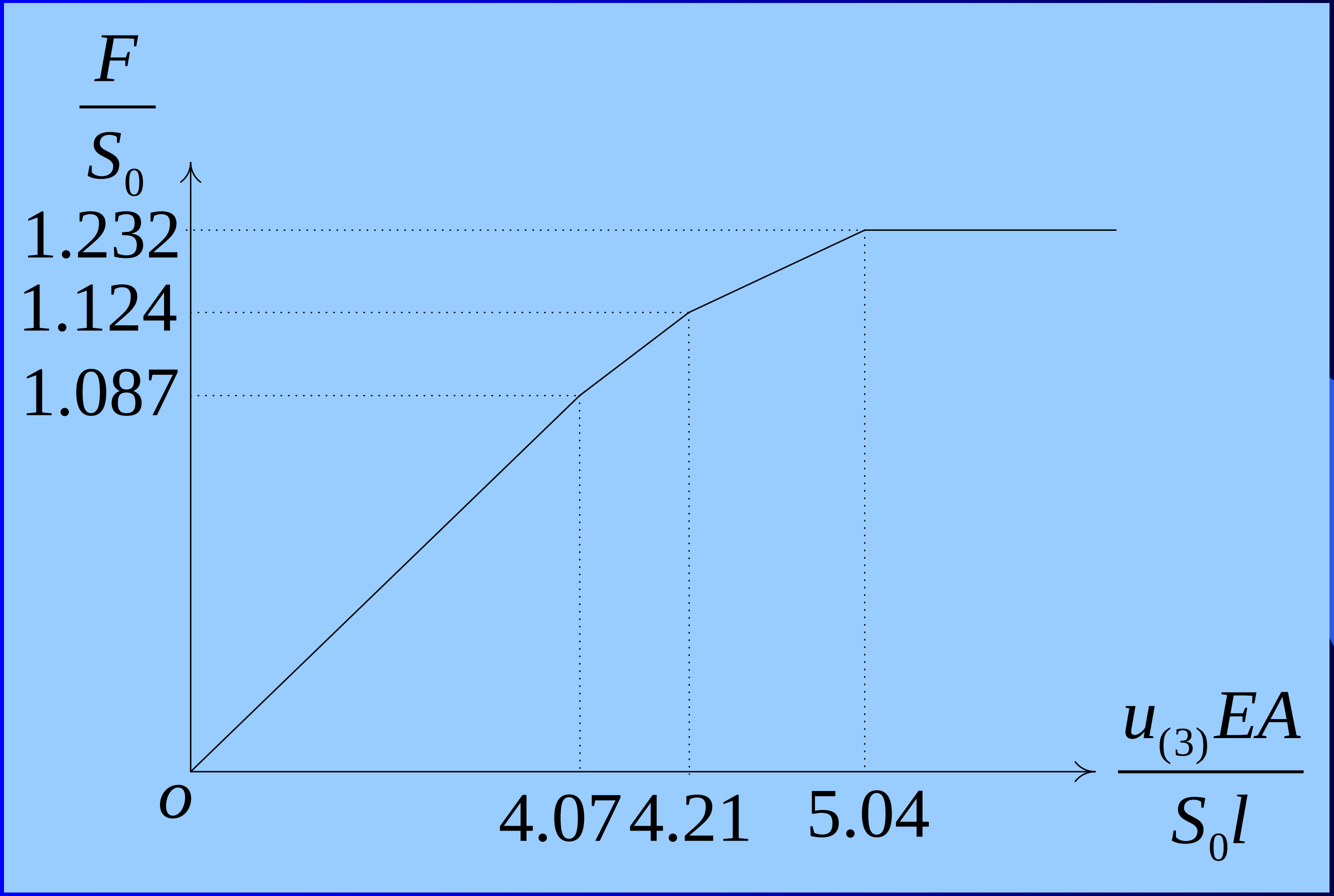
$$\begin{cases}\dot{u} = +13.704\,\dot{F}/K \\ \dot{v} = -3.333\,\dot{F}/K\end{cases}$$

$$\dot{\mathbf{s}} = \mathbf{D}_{ep}^{(4)}\mathbf{B}\dot{\mathbf{d}}$$

$$\begin{cases}\dot{S}_1 = 1.667\dot{F}\,(S_1\dot{S}_1 > 0 \rightarrow \text{Plastic Loading}) \\ \dot{S}_2 = -1.332\dot{F} \\ \dot{S}_3 = 0\end{cases}$$



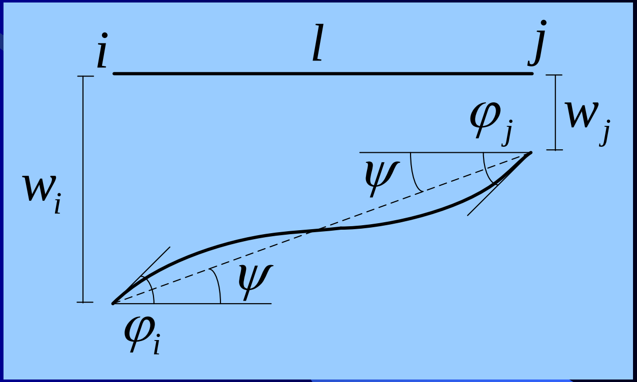
Example, continue…:



Incremental Analysis for Beams:

$$\begin{cases} M_{ij} = \dfrac{2EI}{l}\left(2\varphi_i + \varphi_j - 3\psi\right) \\ M_{ji} = \dfrac{2EI}{l}\left(2\varphi_j + \varphi_i - 3\psi\right) \end{cases}$$

$$\psi = \frac{w_i - w_j}{l}$$



$$\begin{cases} Z_{ij} = -\dfrac{M_{ij} + M_{ji}}{l} \\ Z_{ij} = +\dfrac{M_{ij} + M_{ji}}{l} \end{cases}$$

$$\begin{cases} \varphi_i = \dfrac{2M_{ij} - M_{ji}}{6EI} l + \psi \\ \varphi_j = \dfrac{2M_{ji} - M_{ij}}{6EI} l + \psi \end{cases}$$

$$\begin{Bmatrix} Z_{ij} \\ M_{ij} \\ Z_{ji} \\ M_{ji} \end{Bmatrix} = \frac{EI}{l^3} \begin{bmatrix} 12 & -6l & -12 & -6l \\ -6l & 4l^2 & 6l & 2l^2 \\ -12 & 6l & 12 & 6l \\ -6l & 2l^2 & 6l & 4l^2 \end{bmatrix} \begin{Bmatrix} w_i \\ \varphi_i \\ w_j \\ \varphi_j \end{Bmatrix}$$

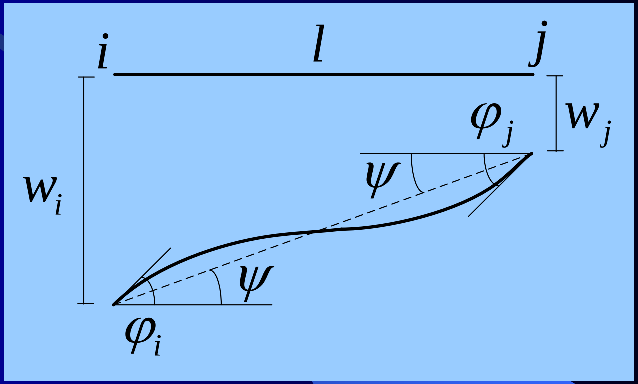
$$\mathbf{s} = \mathbf{DBd}$$

$$\dot{\mathbf{s}} = \mathbf{DB}\dot{\mathbf{d}}$$

Incremental Analysis for Beams:

$$\begin{cases} M_{ij} = 0 \\ M_{ji} = \dfrac{1.5EI}{l}\left(2\varphi_j - 2\psi\right) \end{cases}$$

$$\psi = \frac{w_i - w_j}{l}$$



$$\begin{cases} Z_{ij} = -\dfrac{M_{ji}}{l} \\ Z_{ij} = +\dfrac{M_{ji}}{l} \end{cases}$$

$$\begin{Bmatrix} Z_{ij} \\ M_{ij} \\ Z_{ji} \\ M_{ji} \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 3 & 0 & -3 & -3l \\ 0 & 0 & 0 & 0 \\ -3 & 0 & 3 & 3l \\ -3l & 0 & 3l & 3l^2 \end{bmatrix}\begin{Bmatrix} w_i \\ \varphi_i \\ w_j \\ \varphi_j \end{Bmatrix}$$

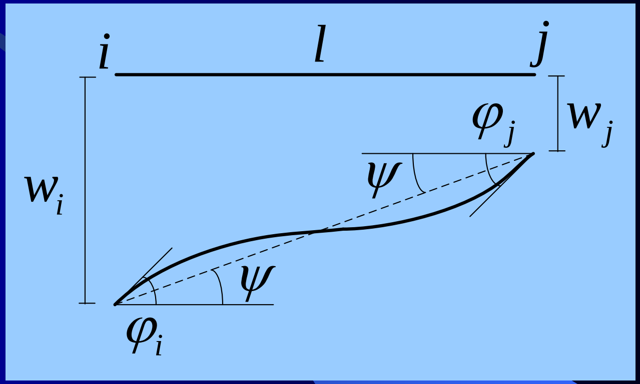
$$\mathbf{s} = \mathbf{DBd}$$

$$\dot{\mathbf{s}} = \mathbf{DB\dot{d}}$$

Incremental Analysis for Beams:

$$\begin{cases} M_{ij} = \dfrac{1.5EI}{l}\left(2\varphi_i - 2\psi\right) \\ M_{ji} = 0 \end{cases}$$

$$\psi = \frac{w_i - w_j}{l}$$



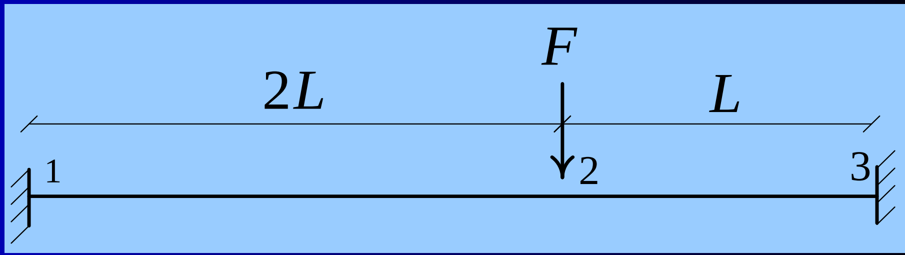
$$\begin{cases} Z_{ij} = -\dfrac{M_{ij}}{l} \\ Z_{ij} = +\dfrac{M_{ij}}{l} \end{cases}$$

$$\begin{Bmatrix} Z_{ij} \\ M_{ij} \\ Z_{ji} \\ M_{ji} \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 3 & -3l & -3 & 0 \\ -3l & 3l^2 & 3l & 0 \\ -3 & 3l & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} w_i \\ \varphi_i \\ w_j \\ \varphi_j \end{Bmatrix}$$

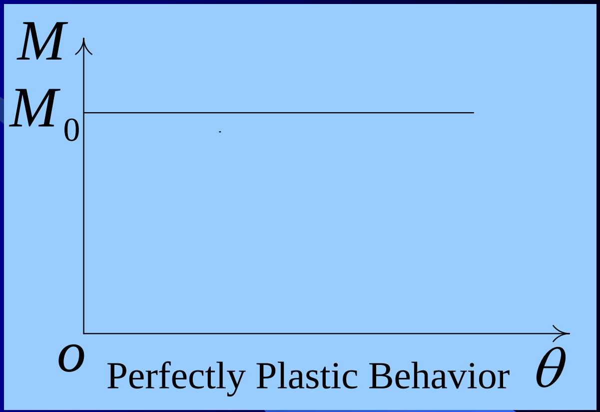
$$\mathbf{s} = \mathbf{DBd}$$

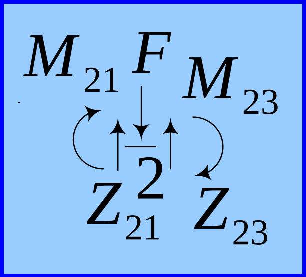
$$\dot{\mathbf{s}} = \mathbf{DB}\dot{\mathbf{d}}$$

Incremental Analysis for Beams, Example:



$$Degree\ of\ Freedom: 2 \qquad \begin{Bmatrix} w_2 \\ \varphi_2 \end{Bmatrix}$$

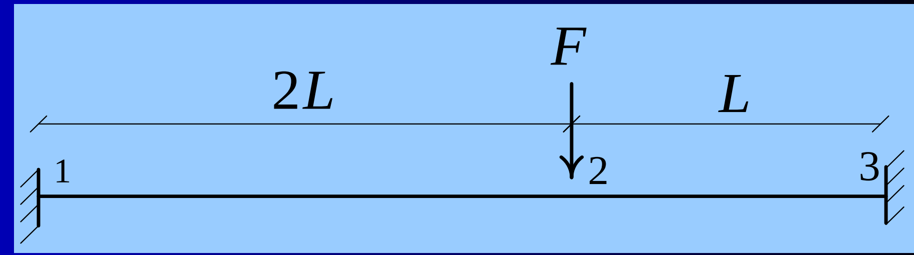




$$Equilibrium\ Equations: \begin{cases} \dot{Z}_{21} + \dot{Z}_{23} = \dot{F} \\ \dot{M}_{21} + \dot{M}_{23} = 0 \end{cases}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{Z}_{21} \\ \dot{M}_{21} \\ \dot{Z}_{23} \\ \dot{M}_{23} \end{Bmatrix} = \begin{Bmatrix} \dot{F} \\ 0 \end{Bmatrix} \qquad \mathbf{B}^T \dot{\mathbf{s}} = \dot{\mathbf{f}}$$

Step 1:



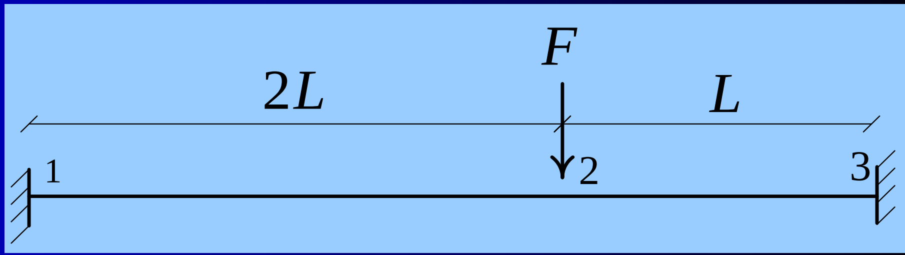
$$\begin{Bmatrix} \dot{Z}_{12} \\ \dot{M}_{12} \\ \dot{Z}_{21} \\ \dot{M}_{21} \end{Bmatrix} = \frac{EI}{8l^3} \begin{bmatrix} 12 & -12l & -12 & -12l \\ -12l & 16l^2 & 12l & 8l^2 \\ -12 & 12l & 12 & 12l \\ -12l & 8l^2 & 12l & 16l^2 \end{bmatrix} \begin{Bmatrix} \cancel{\dot{w}_1} \\ \cancel{\dot{\varphi}_1} \\ \dot{w}_2 \\ \dot{\varphi}_2 \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{Z}_{23} \\ \dot{M}_{23} \\ \dot{Z}_{32} \\ \dot{M}_{32} \end{Bmatrix} = \frac{EI}{l^3} \begin{bmatrix} 12 & -6l & -12 & -6l \\ -6l & 4l^2 & 6l & 2l^2 \\ -12 & 6l & 12 & 6l \\ -6l & 2l^2 & 6l & 4l^2 \end{bmatrix} \begin{Bmatrix} \dot{w}_2 \\ \dot{\varphi}_2 \\ \cancel{\dot{w}_3} \\ \cancel{\dot{\varphi}_3} \end{Bmatrix}$$

$$Equilibrium\ Equations: \begin{cases} \dot{Z}_{21} + \dot{Z}_{23} = \dot{F} \\ \dot{M}_{21} + \dot{M}_{23} = 0 \end{cases}$$

$$\frac{EI}{l^3} \begin{bmatrix} 13.5 & -4.5l \\ -4.5l & 6l^2 \end{bmatrix} \begin{Bmatrix} \dot{w}_2 \\ \dot{\varphi}_2 \end{Bmatrix} = \begin{Bmatrix} \dot{F} \\ 0 \end{Bmatrix}$$

Step 1, continue…:



$$\frac{EI}{l^3}\begin{bmatrix} 13.5 & -4.5l \\ -4.5l & 13.5l^2 \end{bmatrix}\begin{Bmatrix} \dot{w}_2 \\ \dot{\varphi}_2 \end{Bmatrix} = \begin{Bmatrix} \dot{F} \\ 0 \end{Bmatrix}$$

$$\mathbf{K}^{(1)}\dot{\mathbf{d}} = \dot{\mathbf{f}}$$

$$\begin{Bmatrix} \dot{w}_2 \\ \dot{\varphi}_2 \end{Bmatrix}^{(1)} = \begin{Bmatrix} 0.09877l \\ 0.07407 \end{Bmatrix}\frac{\dot{F}l^2}{EI}$$

$$\begin{Bmatrix} \dot{M}_{12} \\ \dot{M}_{21} \\ \dot{M}_{23} \\ \dot{M}_{32} \end{Bmatrix}^{(1)} = \begin{Bmatrix} 0.2222 \\ 0.2963 \\ -0.2963 \\ -0.4444 \end{Bmatrix}\dot{F}l$$

Yields

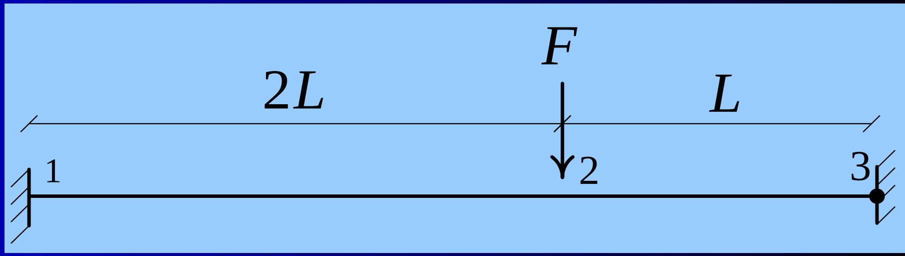
$$\dot{F}^{(1)} = \frac{M_0}{0.4444l} = 2.25\frac{M_0}{l}$$

$$F^{(1)} = 0 + 2.25\frac{M_0}{l} = 2.25\frac{M_0}{l}$$

$$\begin{Bmatrix} w_2 \\ \varphi_2 \end{Bmatrix}^{(1)} = \begin{Bmatrix} 0.2222l \\ 0.1667 \end{Bmatrix}\frac{M_0 l}{EI}$$

$$\begin{Bmatrix} M_{12} \\ M_{21} \\ M_{23} \\ M_{32} \end{Bmatrix}^{(1)} = \begin{Bmatrix} 0.5 \\ 0.6667 \\ -0.6667 \\ -1 \end{Bmatrix}M_0$$

Step 2:



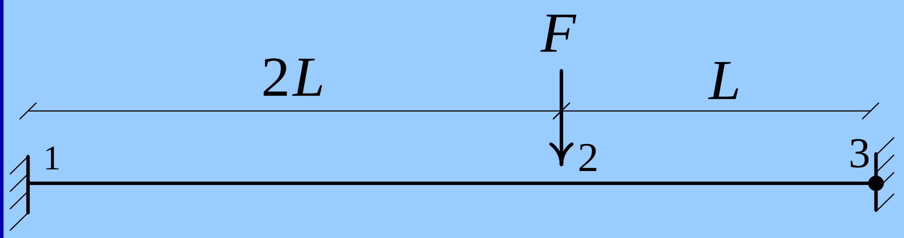
$$\begin{Bmatrix} \dot{Z}_{12} \\ \dot{M}_{12} \\ \dot{Z}_{21} \\ \dot{M}_{21} \end{Bmatrix} = \frac{EI}{8l^3} \begin{bmatrix} 12 & -12l & -12 & -12l \\ -12l & 16l^2 & 12l & 8l^2 \\ -12 & 12l & 12 & 12l \\ -12l & 8l^2 & 12l & 16l^2 \end{bmatrix} \begin{Bmatrix} \cancel{\dot{w}_1} \\ \cancel{\dot{\varphi}_1} \\ \dot{w}_2 \\ \dot{\varphi}_2 \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{Z}_{23} \\ \dot{M}_{23} \\ \dot{Z}_{32} \\ \dot{M}_{32} \end{Bmatrix} = \frac{EI}{l^3} \begin{bmatrix} 3 & -3l & -3 & 0 \\ -3l & 3l^2 & 3l & 0 \\ -3 & 3l & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{w}_2 \\ \dot{\varphi}_2 \\ \cancel{\dot{w}_3} \\ \cancel{\dot{\varphi}_3} \end{Bmatrix}$$

$$Equilibrium\ Equations: \begin{cases} \dot{Z}_{21} + \dot{Z}_{23} = \dot{F} \\ \dot{M}_{21} + \dot{M}_{23} = 0 \end{cases}$$

$$\frac{EI}{l^3} \begin{bmatrix} 4.5 & -1.5l \\ -1.5l & 5l^2 \end{bmatrix} \begin{Bmatrix} \dot{w}_2 \\ \dot{\varphi}_2 \end{Bmatrix} = \begin{Bmatrix} \dot{F} \\ 0 \end{Bmatrix}$$

Step 2, continue…:



$$\frac{EI}{l^3}\begin{bmatrix} 4.5 & -1.5l \\ -1.5l & 5l^2 \end{bmatrix}\begin{Bmatrix} \dot{w}_2 \\ \dot{\varphi}_2 \end{Bmatrix} = \begin{Bmatrix} \dot{F} \\ 0 \end{Bmatrix}$$

$$\mathbf{K}^{(2)}\dot{\mathbf{d}} = \dot{\mathbf{f}}$$

$$\begin{Bmatrix} \dot{w}_2 \\ \dot{\varphi}_2 \end{Bmatrix}^{(2)} = \begin{Bmatrix} 0.24691l \\ 0.07407 \end{Bmatrix}\frac{\dot{F}l^2}{EI}$$

$$\begin{Bmatrix} \dot{M}_{12} \\ \dot{M}_{21} \\ \dot{M}_{23} \\ \dot{M}_{32} \end{Bmatrix}^{(2)} = \begin{Bmatrix} 0.4444 \\ 0.5185 \\ -0.5185 \\ 0 \end{Bmatrix}\dot{F}l$$

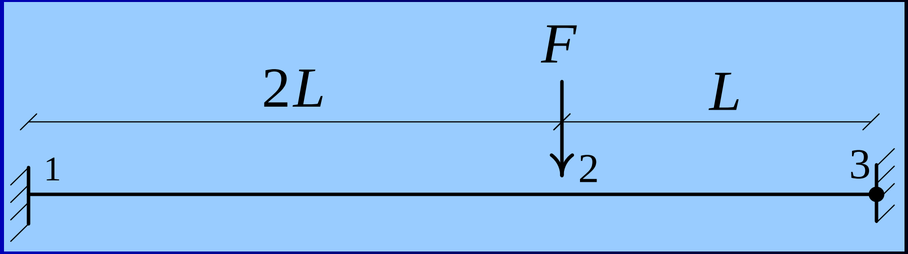
$$\begin{matrix} .5+.4444 \\ .6667+.5185 \end{matrix}$$

Yields

$$\dot{F}^{(2)} = \frac{1-0.6667}{0.5185l}M_0 = 0.6429\frac{M_0}{l}$$

$$F^{(2)} = (2.25+0.6429)\frac{M_0}{l} = 2.8929\frac{M_0}{l}$$

Step 2, continue…:



$$\begin{Bmatrix} M_{12} \\ M_{21} \\ M_{23} \\ M_{32} \end{Bmatrix}^{(2)} = \begin{Bmatrix} 0.7857 \\ 1 \\ -1 \\ -1 \end{Bmatrix} M_0$$

$$\begin{Bmatrix} w_2 \\ \varphi_2 \end{Bmatrix}^{(2)} = \begin{Bmatrix} 0.3809l \\ 0.2143 \end{Bmatrix} \frac{M_0 l}{EI}$$

$$\dot{\varphi}_{3L}^{(2)} = \frac{2\dot{M}_{32}^{(2)} - \dot{M}_{23}^{(2)}}{6EI} l + \frac{\dot{w}_2^{(2)} - \dot{w}_3^{(2)}}{l}$$

$$\dot{\varphi}_{3L}^{(2)} = (-.1111 + 0.1587)\frac{M_0 l}{EI} = 0.0476\frac{M_0 l}{EI}$$

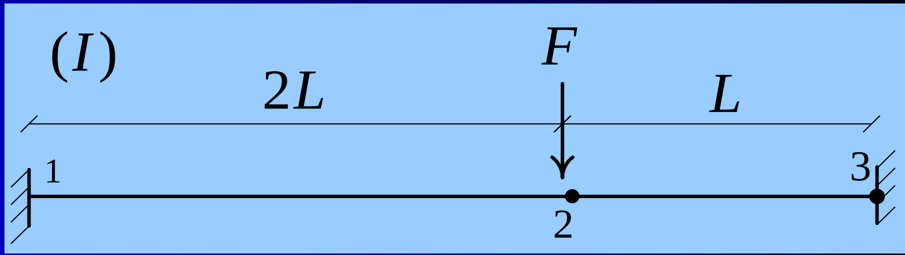
$$\dot{\theta}_3^{(2)} = \dot{\varphi}_{3R}^{(2)} - \dot{\varphi}_{3L}^{(2)}$$

$$\dot{\theta}_3^{(2)} = (0 - 0.0476)\frac{M_0 l}{EI} = -0.0476\frac{M_0 l}{EI}$$

$$\theta_3^{(2)} = (0 - 0.0476)\frac{M_0 l}{EI} = -0.0476\frac{M_0 l}{EI}$$

Step 3:

(Alternative I):



$$\begin{Bmatrix} \dot{Z}_{12} \\ \dot{M}_{12} \\ \dot{Z}_{21} \\ \dot{M}_{21} \end{Bmatrix} = \frac{EI}{8l^3}\begin{bmatrix} 12 & -12l & -12 & -12l \\ -12l & 16l^2 & 12l & 8l^2 \\ -12 & 12l & 12 & 12l \\ -12l & 8l^2 & 12l & 16l^2 \end{bmatrix}\begin{Bmatrix} \cancel{\dot{w}_1} \\ \cancel{\dot{\varphi}_1} \\ \dot{w}_2 \\ \dot{\varphi}_{2L} \end{Bmatrix}$$

$$Equilibrium\ Equations: \begin{cases} \dot{Z}_{21} = \dot{F} \\ \dot{M}_{21} = 0 \end{cases}$$

$$\frac{EI}{l^3}\begin{bmatrix} 1.5 & 1.5l \\ 1.5l & 2l^2 \end{bmatrix}\begin{Bmatrix} \dot{w}_2 \\ \dot{\varphi}_{2L} \end{Bmatrix} = \begin{Bmatrix} \dot{F} \\ 0 \end{Bmatrix}$$

$$\mathbf{K}^{(3)}\dot{\mathbf{d}} = \dot{\mathbf{f}}$$

$$\begin{Bmatrix} \dot{w}_2 \\ \dot{\varphi}_{2L} \end{Bmatrix}^{(3)} = \begin{Bmatrix} 2.66667l \\ -2 \end{Bmatrix}\frac{\dot{F}l^2}{EI}$$

$$\begin{Bmatrix} \dot{M}_{12} \\ \dot{M}_{21} \\ \dot{M}_{23} \\ \dot{M}_{32} \end{Bmatrix}^{(3)} = \begin{Bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{Bmatrix}\dot{F}l$$

Yields

Step 3, continue…:

$$\dot{F}^{(3)} = \frac{1-0.7857}{2l}M_0 = 0.1072\frac{M_0}{l}$$

$$F^{(3)} = (2.8929+0.1073)\frac{M_0}{l} = 3\frac{M_0}{l}$$

$$\begin{Bmatrix} M_{12} \\ M_{21} \\ M_{23} \\ M_{32} \end{Bmatrix}^{(3)} = \begin{Bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{Bmatrix} M_0$$

$$\begin{Bmatrix} w_2 \\ \varphi_{2L} \end{Bmatrix}^{(3)} = \begin{Bmatrix} 0.6667l \\ 0 \end{Bmatrix} \frac{M_0 l}{EI}$$

$$\dot{\varphi}_{3L}^{(3)} = \frac{2\dot{M}_{32}^{(3)} - \dot{M}_{23}^{(3)}}{6EI} l + \frac{\dot{w}_2^{(3)} - \dot{w}_3^{(3)}}{l}$$

$$\dot{\varphi}_{3L}^{(3)} = (0+0.2859)\frac{M_0 l}{EI} = 0.2859\frac{M_0 l}{EI}$$

$$\dot{\theta}_3^{(3)} = \dot{\varphi}_{3R}^{(3)} - \dot{\varphi}_{3L}^{(3)}$$

$$\dot{\theta}_3^{(3)} = (0-0.2859)\frac{M_0 l}{EI} = -0.2859\frac{M_0 l}{EI}$$

$$\theta_3^{(3)} = (-0.0476-0.2859)\frac{M_0 l}{EI} = -0.3333\frac{M_0 l}{EI}$$

Step 3, continue…:

$$\dot{\varphi}_{2R}^{(3)} = \frac{2\dot{M}_{23}^{(3)} - \dot{M}_{32}^{(3)}}{6EI} l + \frac{\dot{w}_2^{(3)} - \dot{w}_3^{(3)}}{l}$$

$$\dot{\varphi}_{2R}^{(3)} = (0 + 0.2859)\frac{M_0 l}{EI} = 0.2859\frac{M_0 l}{EI}$$

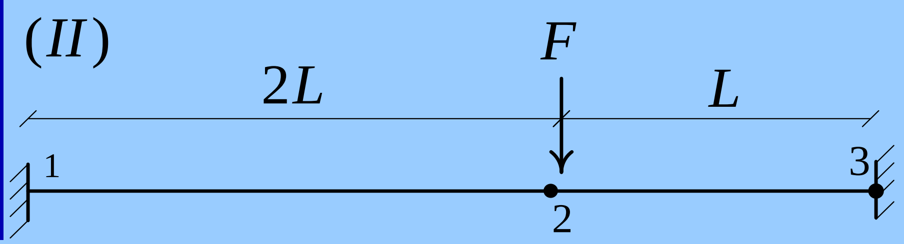
$$\dot{\theta}_2^{(3)} = \dot{\varphi}_{2R}^{(3)} - \dot{\varphi}_{2L}^{(3)}$$

$$\dot{\theta}_2^{(3)} = (0.2859 - (-0.2144))\frac{M_0 l}{EI} = 0.5\frac{M_0 l}{EI}$$

$$\theta_2^{(3)} = (0 + 0.5)\frac{M_0 l}{EI} = +0.5\frac{M_0 l}{EI}$$

Step 3:

(Alternative II):



$$\begin{Bmatrix} \dot{Z}_{12} \\ \dot{M}_{12} \\ \dot{Z}_{21} \\ \dot{M}_{21} \end{Bmatrix} = \frac{EI}{8l^3}\begin{bmatrix} 3 & -6l & -3 & 0 \\ -6l & 12l^2 & 6l & 0 \\ -3 & 6l & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \cancel{\dot{w}_1} \\ \cancel{\dot{\varphi}_1} \\ \dot{w}_2 \\ \dot{\varphi}_{2L} \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{Z}_{23} \\ \dot{M}_{23} \\ \dot{Z}_{32} \\ \dot{M}_{32} \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 3 & -3l & -3 & 0 \\ -3l & 3l^2 & 3l & 0 \\ -3 & 3l & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \dot{w}_2 \\ \dot{\varphi}_{2R} \\ \cancel{\dot{w}_3} \\ \cancel{\dot{\varphi}_3} \end{Bmatrix}$$

$$Equilibrium\ Equations: \begin{cases} \dot{Z}_{21} + \dot{Z}_{23} = \dot{F} \\ \dot{M}_{23} = 0 \end{cases}$$

$$\frac{EI}{l^3}\begin{bmatrix} 3.375 & -3l \\ -3l & 3l^2 \end{bmatrix}\begin{Bmatrix} \dot{w}_2 \\ \dot{\varphi}_{2R} \end{Bmatrix} = \begin{Bmatrix} \dot{F} \\ 0 \end{Bmatrix}$$

Step 3, continue…:

$$\frac{EI}{l^3}\begin{bmatrix} 3.375 & -3l \\ -3l & 3l^2 \end{bmatrix}\begin{Bmatrix} \dot{w}_2 \\ \dot{\varphi}_{2R} \end{Bmatrix} = \begin{Bmatrix} \dot{F} \\ 0 \end{Bmatrix}$$

$$\mathbf{K}^{(3)}\dot{\mathbf{d}} = \dot{\mathbf{f}}$$

$$\begin{Bmatrix} \dot{w}_2 \\ \dot{\varphi}_{2R} \end{Bmatrix}^{(3)} = \begin{Bmatrix} 2.66667l \\ 2.66667 \end{Bmatrix}\frac{\dot{F}l^2}{EI}$$

$$\begin{Bmatrix} \dot{M}_{12} \\ \dot{M}_{21} \\ \dot{M}_{23} \\ \dot{M}_{32} \end{Bmatrix}^{(3)} = \begin{Bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{Bmatrix}\dot{F}l$$

Yields

$$\dot{F}^{(3)} = \frac{1-0.7857}{2l}M_0 = 0.1072\frac{M_0}{l}$$

$$F^{(3)} = (2.8929 + 0.1073)\frac{M_0}{l} = 3\frac{M_0}{l}$$

$$\begin{Bmatrix} w_2 \\ \varphi_{2R} \end{Bmatrix}^{(3)} = \begin{Bmatrix} 0.6667l \\ 0.5 \end{Bmatrix}\frac{M_0 l}{EI}$$

$$\begin{Bmatrix} M_{12} \\ M_{21} \\ M_{23} \\ M_{32} \end{Bmatrix}^{(3)} = \begin{Bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{Bmatrix}M_0$$

Step 3, continue…:

$$\dot{\varphi}_{3L}^{(3)}=\frac{2\dot{M}_{32}^{(3)}-\dot{M}_{23}^{(3)}}{6EI}l+\frac{\dot{w}_{2}^{(3)}-\dot{w}_{3}^{(3)}}{l}$$

$$\dot{\varphi}_{3L}^{(3)}=\left(0+0.2859\right)\frac{M_0 l}{EI}=0.2859\frac{M_0 l}{EI}$$

$$\dot{\theta}_{3}^{(3)}=\dot{\varphi}_{3R}^{(3)}-\dot{\varphi}_{3L}^{(3)}$$

$$\dot{\theta}_{3}^{(3)}=\left(0-0.2859\right)\frac{M_0 l}{EI}=-0.2859\frac{M_0 l}{EI}$$

$$\theta_{3}^{(3)}=\left(-0.0476-0.2859\right)\frac{M_0 l}{EI}=-0.3333\frac{M_0 l}{EI}$$

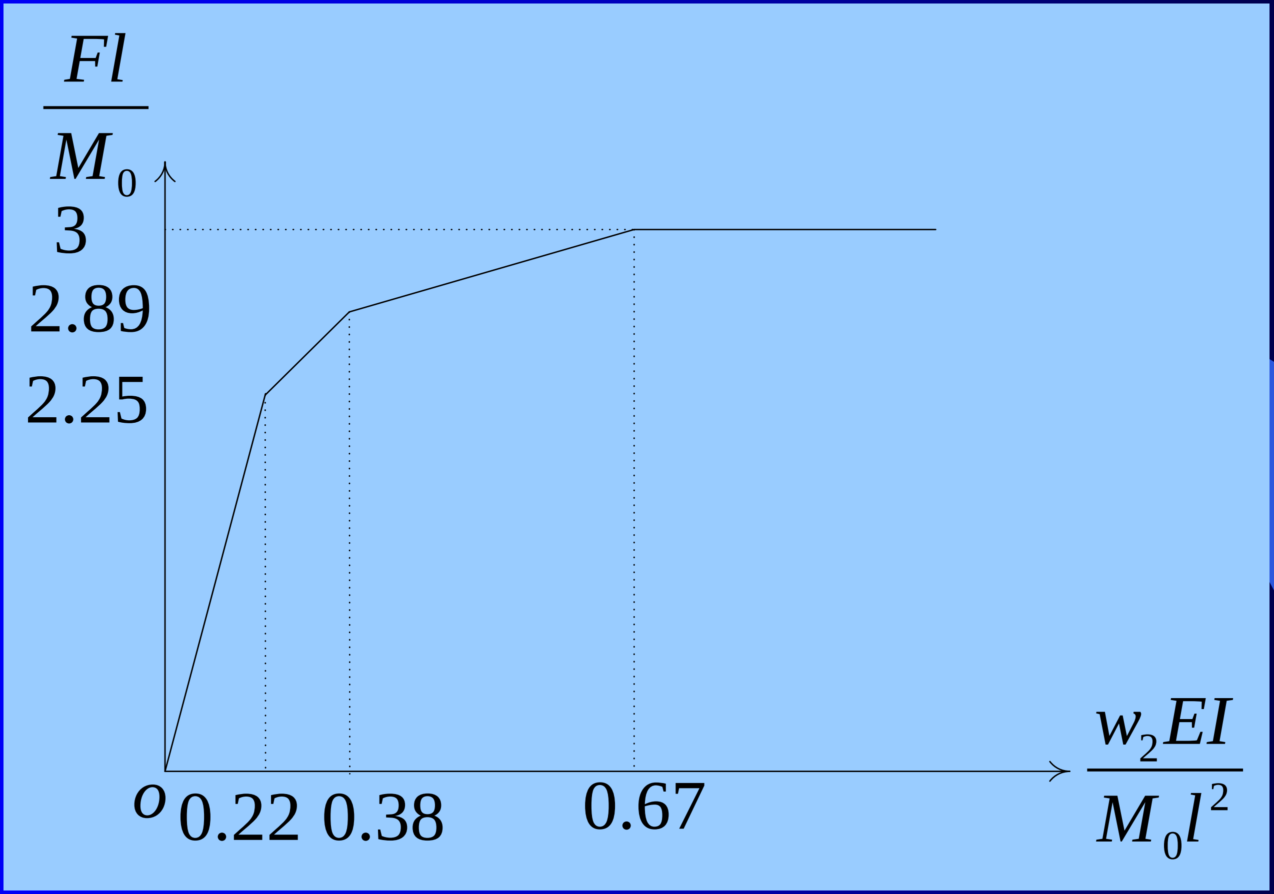
$$\dot{\varphi}_{2L}^{(3)}=\frac{2\dot{M}_{21}^{(3)}-\dot{M}_{12}^{(3)}}{6EI}2l+\frac{\dot{w}_{1}^{(3)}-\dot{w}_{2}^{(3)}}{2l}$$

$$\dot{\varphi}_{2L}^{(3)}=\left(-0.0715-0.1429\right)\frac{M_0 l}{EI}=-0.2144\frac{M_0 l}{EI}$$

$$\dot{\theta}_{2}^{(3)}=\dot{\varphi}_{2R}^{(3)}-\dot{\varphi}_{2L}^{(3)}$$

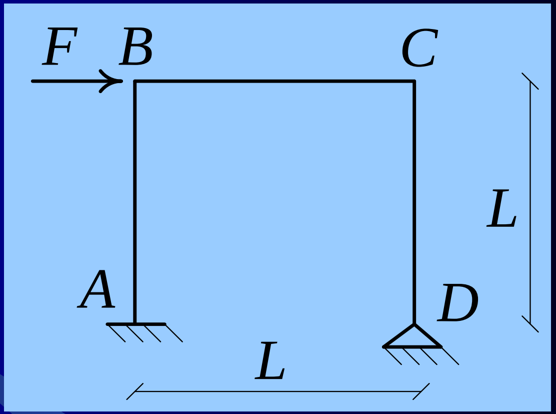
$$\theta_{2}^{(3)}=\left(0+0.5\right)\frac{M_0 l}{EI}=0.5\frac{M_0 l}{EI}$$

Example, continue…:

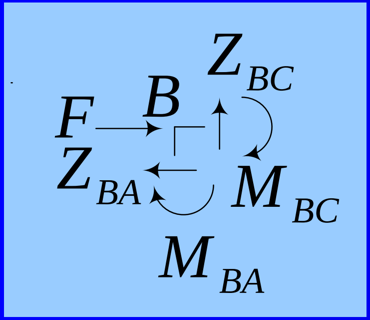


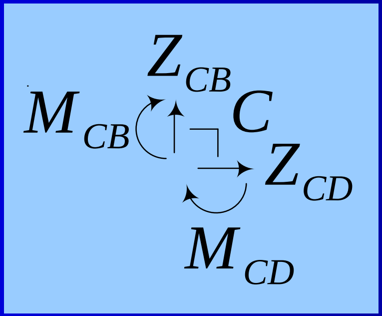
Incremental Analysis for Frames, Example:

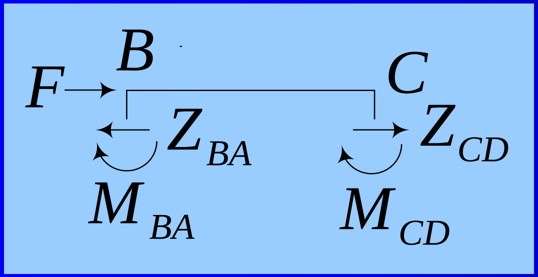
$$Degree\ of\ Freedom: 3 \qquad \begin{cases} u_B = u_C \\ \varphi_B \\ \varphi_C \end{cases}$$

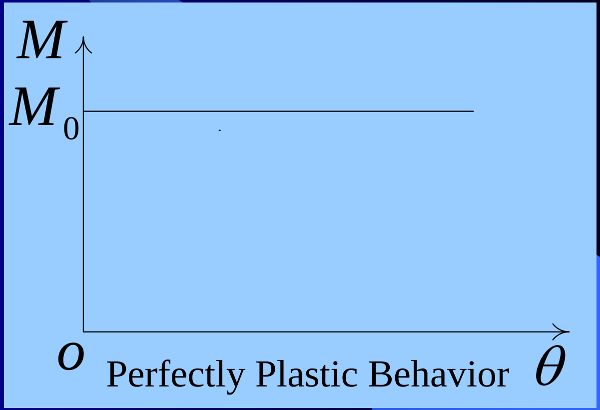


$$Free\ Diagrams:$$









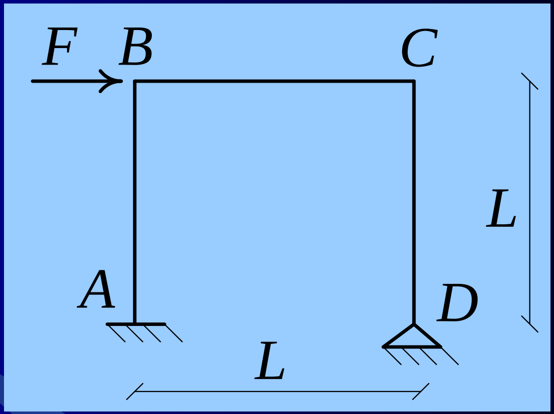
$$Equilibrium\ Equations: \begin{cases} \dot{Z}_{BA} - \dot{Z}_{CD} = \dot{F} \\ \dot{M}_{BA} + \dot{M}_{BC} = 0 \\ \dot{M}_{CD} + \dot{M}_{CB} = 0 \end{cases}$$

Step 1:

$$\begin{Bmatrix} \dot{Z}_{AB} \\ \dot{M}_{AB} \\ \dot{Z}_{BA} \\ \dot{M}_{BA} \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 12 & -6l & -12 & -6l \\ -6l & 4l^2 & 6l & 2l^2 \\ -12 & 6l & 12 & 6l \\ -6l & 2l^2 & 6l & 4l^2 \end{bmatrix}\begin{Bmatrix} \cancel{\dot{w}_A} \\ \cancel{\dot{\varphi}_A} \\ \overrightarrow{\dot{w}_B} \\ \dot{\varphi}_B \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{Z}_{BC} \\ \dot{M}_{BC} \\ \dot{Z}_{CB} \\ \dot{M}_{CB} \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 12 & -6l & -12 & -6l \\ -6l & \boxed{4l^2} & 6l & \boxed{2l^2} \\ -12 & 6l & 12 & 6l \\ -6l & \boxed{2l^2} & 6l & \boxed{4l^2} \end{bmatrix}\begin{Bmatrix} \downarrow\cancel{\dot{w}_B} \\ \dot{\varphi}_B \\ \downarrow\cancel{\dot{w}_C} \\ \dot{\varphi}_C \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{Z}_{CD} \\ \dot{M}_{CD} \\ \dot{Z}_{DC} \\ \dot{M}_{DC} \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 3 & -3l & -3 & 0 \\ -3l & 3l^2 & 3l & 0 \\ -3 & 3l & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \overleftarrow{\dot{w}_C} \\ \dot{\varphi}_C \\ \cancel{\dot{w}_D} \\ \cancel{\dot{\varphi}_D} \end{Bmatrix}$$



$$\overrightarrow{\dot{w}_B} = -\overleftarrow{\dot{w}_C} = \dot{u}$$

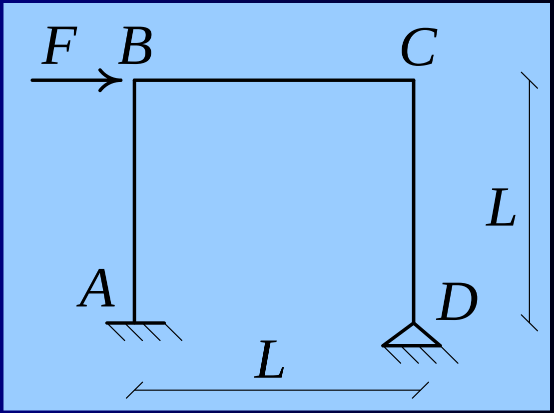
$$\begin{cases} \dot{Z}_{BA} - \dot{Z}_{CD} = \dot{F} \\ \dot{M}_{BA} + \dot{M}_{BC} = 0 \\ \dot{M}_{CD} + \dot{M}_{CB} = 0 \end{cases}$$

$$\frac{EI}{l^3}\begin{bmatrix} 15 & 6l & 3l \\ 6l & 8l^2 & 2l^2 \\ 3l & 2l^2 & 7l^2 \end{bmatrix}\begin{Bmatrix} \dot{u} \\ \dot{\varphi}_B \\ \dot{\varphi}_C \end{Bmatrix} = \begin{Bmatrix} \dot{F} \\ 0 \\ 0 \end{Bmatrix}$$

Step 1, continue…:

$$\frac{EI}{l^3}\begin{bmatrix}15 & 6l & 3l\\ 6l & 8l^2 & 2l^2\\ 3l & 2l^2 & 7l^2\end{bmatrix}\begin{Bmatrix}\dot{u}\\ \dot{\varphi}_B\\ \dot{\varphi}_C\end{Bmatrix}=\begin{Bmatrix}\dot{F}\\ 0\\ 0\end{Bmatrix}$$

$$\mathbf{K}^{(1)}\dot{\mathbf{d}}=\dot{\mathbf{f}}$$



$$\begin{Bmatrix}\dot{u}\\ \dot{\varphi}_B\\ \dot{\varphi}_C\end{Bmatrix}=\begin{Bmatrix}0.0985l\\ -0.0682\\ -0.0227\end{Bmatrix}\frac{\dot{F}l^2}{EI}$$

$$\begin{Bmatrix}\dot{M}_{AB}\\ \dot{M}_{BA}\\ \dot{M}_{BC}\\ \dot{M}_{CB}\\ \dot{M}_{CD}\end{Bmatrix}=\begin{Bmatrix}0.4546\\ 0.3182\\ -0.3182\\ -0.2272\\ 0.2272\end{Bmatrix}\dot{F}l$$

Yields

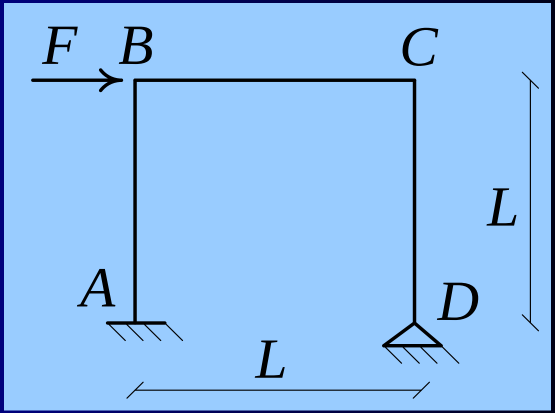
$$\dot{F}^{(1)}=\frac{M_0}{0.4546l}=2.2\frac{M_0}{l}$$

$$F^{(1)}=0+2.2\frac{M_0}{l}=2.2\frac{M_0}{l}$$

Step 1, continue…:

$$\begin{Bmatrix} u \\ \varphi_B \\ \varphi_C \end{Bmatrix}^{(1)} = \begin{Bmatrix} 0.2167l \\ -0.15 \\ -0.05 \end{Bmatrix} \frac{M_0 l}{EI}$$

$$\begin{Bmatrix} M_{AB} \\ M_{BA} \\ M_{BC} \\ M_{CB} \\ M_{CD} \end{Bmatrix}^{(1)} = \begin{Bmatrix} 1 \\ 0.7 \\ -0.7 \\ -0.5 \\ 0.5 \end{Bmatrix} M_0$$

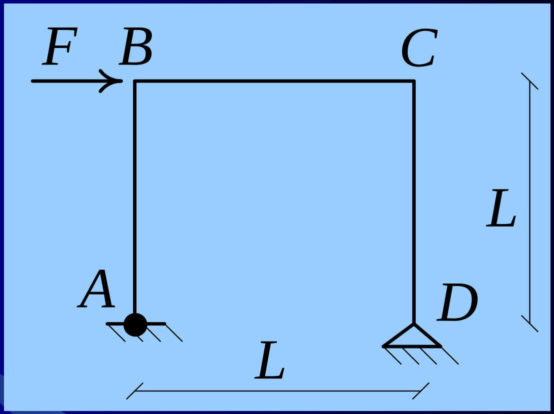


Step 2:

$$\begin{Bmatrix} \dot{Z}_{AB} \\ \dot{M}_{AB} \\ \dot{Z}_{BA} \\ \dot{M}_{BA} \end{Bmatrix} = \frac{EI}{l^3} \begin{bmatrix} 3 & 0 & -3 & -3l \\ 0 & 0 & 0 & 0 \\ -3 & 0 & \boxed{3} & \boxed{3l} \\ -3l & 0 & \boxed{3l} & \boxed{3l^2} \end{bmatrix} \begin{Bmatrix} \cancel{\dot{w}_A} \\ \cancel{\dot{\varphi}_A} \\ \overrightarrow{\dot{w}}_B \\ \dot{\varphi}_B \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{Z}_{BC} \\ \dot{M}_{BC} \\ \dot{Z}_{CB} \\ \dot{M}_{CB} \end{Bmatrix} = \frac{EI}{l^3} \begin{bmatrix} 12 & -6l & -12 & -6l \\ -6l & \boxed{4l^2} & 6l & \boxed{2l^2} \\ -12 & 6l & 12 & 6l \\ -6l & \boxed{2l^2} & 6l & \boxed{4l^2} \end{bmatrix} \begin{Bmatrix} \downarrow \cancel{\dot{w}_B} \\ \dot{\varphi}_B \\ \downarrow \cancel{\dot{w}_C} \\ \dot{\varphi}_C \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{Z}_{CD} \\ \dot{M}_{CD} \\ \dot{Z}_{DC} \\ \dot{M}_{DC} \end{Bmatrix} = \frac{EI}{l^3} \begin{bmatrix} \boxed{3} & \boxed{-3l} & -3 & 0 \\ \boxed{-3l} & \boxed{3l^2} & 3l & 0 \\ -3 & 3l & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \overleftarrow{\dot{w}}_C \\ \dot{\varphi}_C \\ \cancel{\dot{w}_D} \\ \cancel{\dot{\varphi}_D} \end{Bmatrix}$$



$$\overrightarrow{\dot{w}}_B = -\overleftarrow{\dot{w}}_C = \dot{u}$$

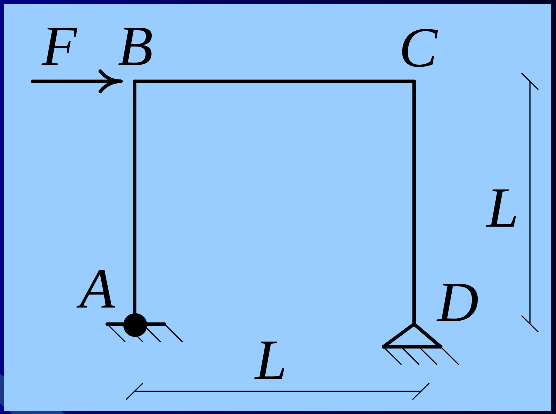
$$\begin{cases} \dot{Z}_{BA} - \dot{Z}_{CD} = \dot{F} \\ \dot{M}_{BA} + \dot{M}_{BC} = 0 \\ \dot{M}_{CD} + \dot{M}_{CB} = 0 \end{cases}$$

$$\frac{EI}{l^3} \begin{bmatrix} 6 & 3l & 3l \\ 3l & 7l^2 & 2l^2 \\ 3l & 2l^2 & 7l^2 \end{bmatrix} \begin{Bmatrix} \dot{u} \\ \dot{\varphi}_B \\ \dot{\varphi}_C \end{Bmatrix} = \begin{Bmatrix} \dot{F} \\ 0 \\ 0 \end{Bmatrix}$$

Step 2, continue…:

$$\frac{EI}{l^3}\begin{bmatrix} 6 & 3l & 3l \\ 3l & 7l^2 & 2l^2 \\ 3l & 2l^2 & 7l^2 \end{bmatrix}\begin{Bmatrix} \dot{u} \\ \dot{\varphi}_B \\ \dot{\varphi}_C \end{Bmatrix} = \begin{Bmatrix} \dot{F} \\ 0 \\ 0 \end{Bmatrix}$$

$$\mathbf{K}^{(2)}\dot{\mathbf{d}} = \dot{\mathbf{f}}$$



$$\begin{Bmatrix} \dot{u} \\ \dot{\varphi}_B \\ \dot{\varphi}_C \end{Bmatrix} = \begin{Bmatrix} 0.25l \\ -0.0833 \\ -0.0833 \end{Bmatrix}\frac{\dot{F}l^2}{EI}$$

$$\begin{Bmatrix} \dot{M}_{AB} \\ \dot{M}_{BA} \\ \dot{M}_{BC} \\ \dot{M}_{CB} \\ \dot{M}_{CD} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.5 \\ -0.5 \\ -0.5 \\ 0.5 \end{Bmatrix}\dot{F}l$$

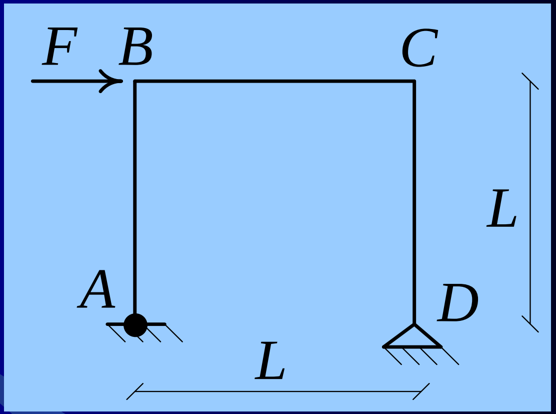
$$\begin{matrix} 1M_0 + 0\dot{F}l \\ 0.7M_0 + 0.5\dot{F}l \\ -0.7M_0 - 0.5\dot{F}l \\ -0.5M_0 - 0.5\dot{F}l \\ 0.5M_0 + 0.5\dot{F}l \end{matrix}$$

Yields

$$\dot{F}^{(2)} = \frac{1-0.7}{0.5l}M_0 = 0.6\frac{M_0}{l}$$

$$F^{(2)} = 2.2\frac{M_0}{l} + 0.6\frac{M_0}{l} = 2.8\frac{M_0}{l}$$

Step 2, continue…:



$$\begin{Bmatrix} u \\ \varphi_B \\ \varphi_C \end{Bmatrix}^{(2)} = \begin{Bmatrix} 0.3667l \\ -0.2 \\ -0.1 \end{Bmatrix} \frac{M_0}{EI}$$

$$\begin{Bmatrix} M_{AB} \\ M_{BA} \\ M_{BC} \\ M_{CB} \\ M_{CD} \end{Bmatrix}^{(2)} = \begin{Bmatrix} 1 \\ 1 \\ -1 \\ -0.8 \\ 0.8 \end{Bmatrix} M_0$$

$$\dot{\varphi}_{AT}^{(2)} = \frac{2\dot{M}_{AB}^{(2)} - \dot{M}_{BA}^{(2)}}{6EI} l + \frac{\dot{w}_A^{(2)} - \dot{w}_B^{(2)}}{l}$$

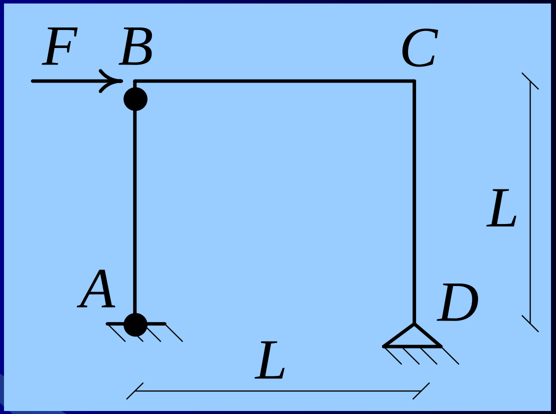
$$\dot{\varphi}_{AT}^{(2)} = (-0.05 - 0.15)\frac{M_0 l}{EI} = -.2\frac{M_0 l}{EI}$$

$$\dot{\theta}_A^{(2)} = \dot{\varphi}_{AT}^{(2)} - \dot{\varphi}_{AB}^{(2)}$$

$$\dot{\theta}_A^{(2)} = (-0.2 - 0)\frac{M_0 l}{EI} = -0.2\frac{M_0 l}{EI}$$

$$\theta_A^{(2)} = (0 + (-0.2))\frac{M_0 l}{EI} = -0.2\frac{M_0 l}{EI}$$

Step 3:



$$\overrightarrow{\dot w}_B = -\overleftarrow{\dot w}_C = \dot u$$

$$\begin{Bmatrix} \dot Z_{BC} \\ \dot M_{BC} \\ \dot Z_{CB} \\ \dot M_{CB} \end{Bmatrix} = \frac{EI}{l^3} \begin{bmatrix} 12 & -6l & -12 & -6l \\ -6l & \boxed{4l^2} & 6l & \boxed{2l^2} \\ -12 & 6l & 12 & 6l \\ -6l & \boxed{2l^2} & 6l & \boxed{4l^2} \end{bmatrix} \begin{Bmatrix} \downarrow \cancel{\dot w_B} \\ \dot\varphi_{BR} \\ \downarrow \cancel{\dot w_C} \\ \dot\varphi_C \end{Bmatrix}$$

$$\begin{cases} \dot Z_{BA} - \dot Z_{CD} = \dot F \\ \dot M_{BA} + \dot M_{BC} = 0 \\ \dot M_{CD} + \dot M_{CB} = 0 \end{cases}$$

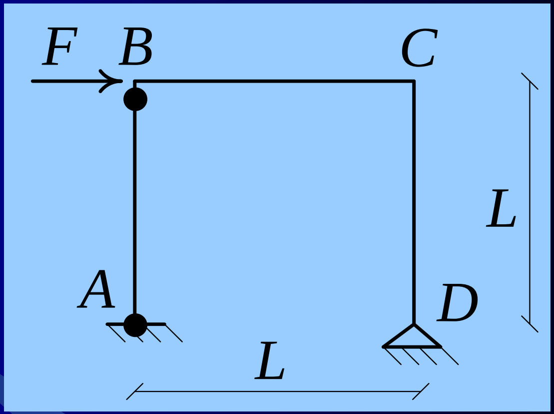
$$\begin{Bmatrix} \dot Z_{CD} \\ \dot M_{CD} \\ \dot Z_{DC} \\ \dot M_{DC} \end{Bmatrix} = \frac{EI}{l^3} \begin{bmatrix} 3 & -3l & -3 & 0 \\ -3l & 3l^2 & 3l & 0 \\ -3 & 3l & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \overleftarrow{\dot w}_C \\ \dot\varphi_C \\ \cancel{\dot w_D} \\ \cancel{\dot\varphi_D} \end{Bmatrix}$$

$$\frac{EI}{l^3} \begin{bmatrix} 3 & 0 & 3l \\ 0 & 4l^2 & 2l^2 \\ 3l & 2l^2 & 7l^2 \end{bmatrix} \begin{Bmatrix} \dot u \\ \dot\varphi_{BR} \\ \dot\varphi_C \end{Bmatrix} = \begin{Bmatrix} \dot F \\ 0 \\ 0 \end{Bmatrix}$$

Step 3, continue…:

$$\frac{EI}{l^3}\begin{bmatrix} 3 & 0 & 3l \\ 0 & 4l^2 & 2l^2 \\ 3l & 2l^2 & 7l^2 \end{bmatrix}\begin{Bmatrix} \dot{u} \\ \dot{\varphi}_{BR} \\ \dot{\varphi}_C \end{Bmatrix} = \begin{Bmatrix} \dot{F} \\ 0 \\ 0 \end{Bmatrix}$$

$$\mathbf{K}^{(3)}\dot{\mathbf{d}} = \dot{\mathbf{f}}$$



$$\begin{Bmatrix} \dot{u} \\ \dot{\varphi}_{BR} \\ \dot{\varphi}_C \end{Bmatrix} = \begin{Bmatrix} 0.6667l \\ 0.1667 \\ -0.3333 \end{Bmatrix}\frac{\dot{F}l^2}{EI}$$

$$\begin{Bmatrix} \dot{M}_{AB} \\ \dot{M}_{BA} \\ \dot{M}_{BC} \\ \dot{M}_{CB} \\ \dot{M}_{CD} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{Bmatrix}\dot{F}l$$

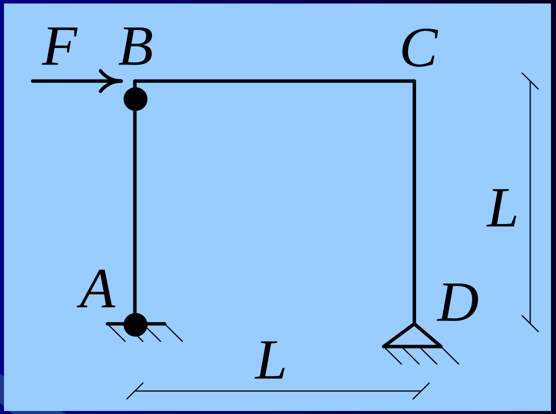
$$\begin{array}{c} 1M_0 + 0\dot{F}l \\ 1M_0 + 0\dot{F}l \\ -1M_0 - 0\dot{F}l \\ -0.8M_0 - 1\dot{F}l \\ 0.8M_0 + 1\dot{F}l \end{array}$$

Yields

$$\dot{F}^{(3)} = \frac{1-0.8}{1l}M_0 = 0.2\frac{M_0}{l}$$

$$F^{(3)} = 2.8\frac{M_0}{l} + 0.2\frac{M_0}{l} = 3\frac{M_0}{l}$$

Step 3, continue…:



$$\begin{Bmatrix} u \\ \varphi_{BR} \\ \varphi_C \end{Bmatrix}^{(3)} = \begin{Bmatrix} 0.5l \\ -0.1667 \\ -0.1667 \end{Bmatrix} \frac{M_0 l}{EI}$$

$$\begin{Bmatrix} M_{AB} \\ M_{BA} \\ M_{BC} \\ M_{CB} \\ M_{CD} \end{Bmatrix}^{(3)} = \begin{Bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 1 \end{Bmatrix} M_0$$

$$\dot{\varphi}_{AT}^{(3)} = \frac{2\dot{M}_{AB}^{(3)} - \dot{M}_{BA}^{(3)}}{6EI} l + \frac{\dot{w}_A^{(3)} - \dot{w}_B^{(3)}}{l}$$

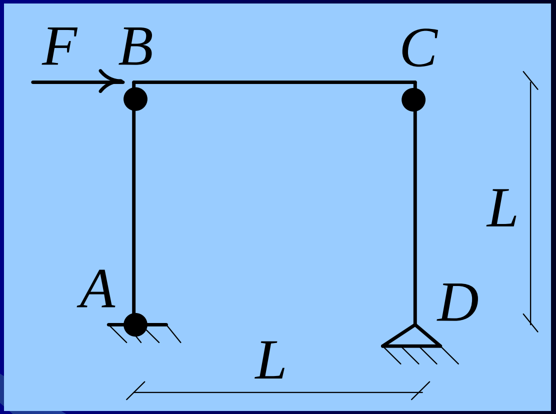
$$\dot{\varphi}_{AT}^{(3)} = (0 - 0.1333)\frac{M_0 l}{EI} = -0.1333 \frac{M_0 l}{EI}$$

$$\dot{\theta}_A^{(3)} = \dot{\varphi}_{AT}^{(3)} - \dot{\varphi}_{AB}^{(3)}$$

$$\dot{\theta}_A^{(3)} = (-0.1333 - 0)\frac{M_0 l}{EI} = -0.1333 \frac{M_0 l}{EI}$$

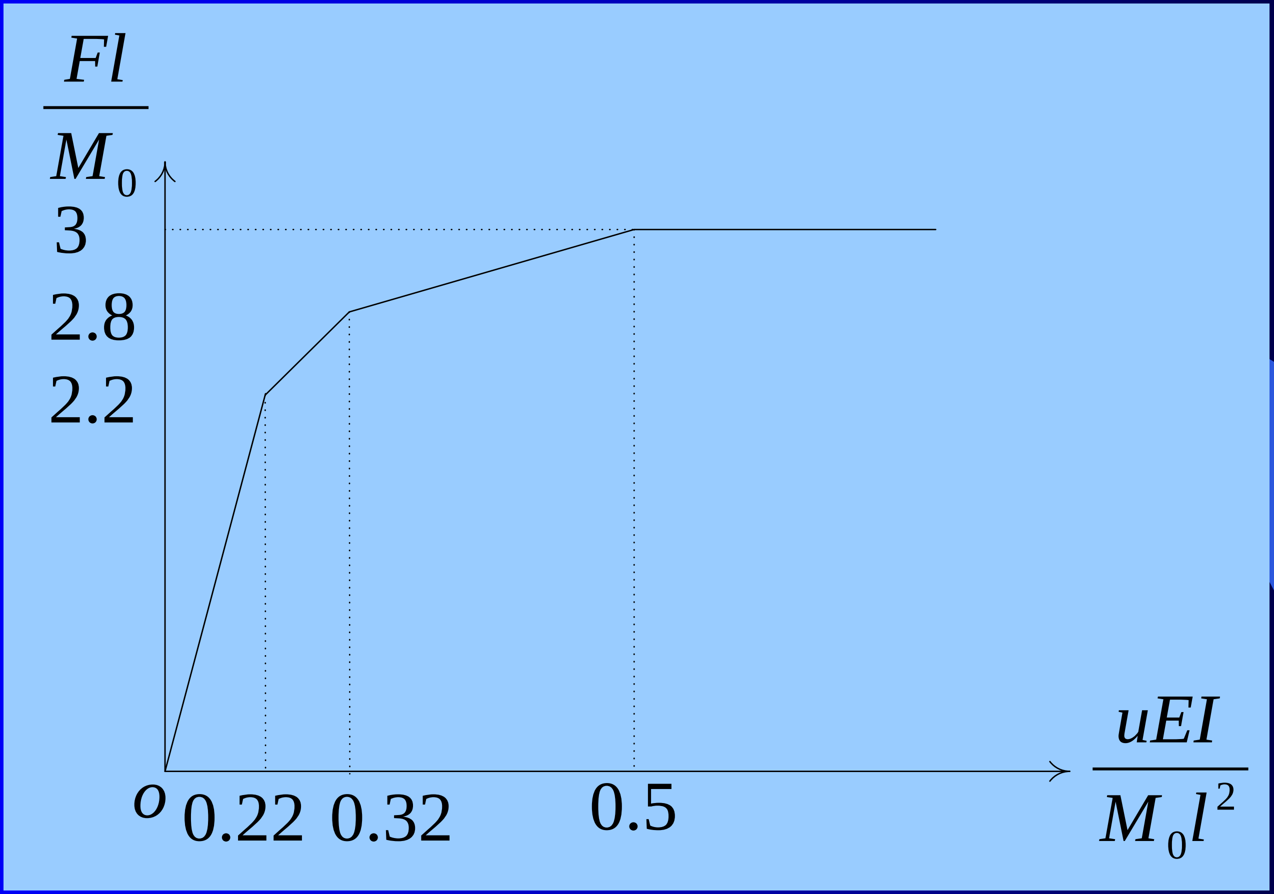
$$\theta_A^{(3)} = (-0.2 - 0.1333)\frac{M_0 l}{EI} = -0.3333 \frac{M_0 l}{EI}$$

Step 4:



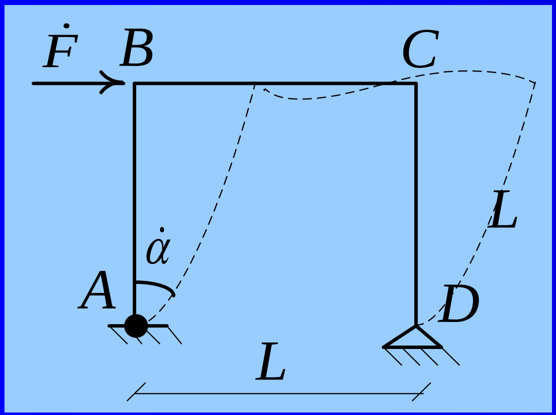
Unstable

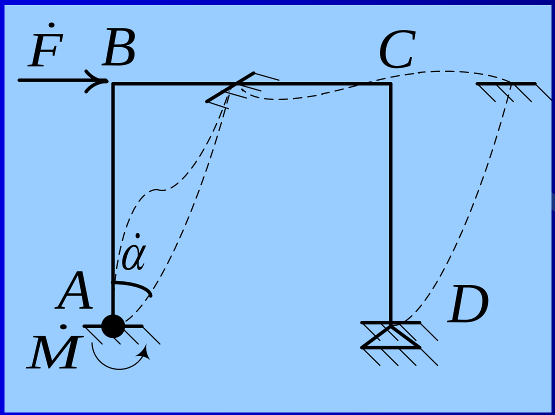
Example, continue…:

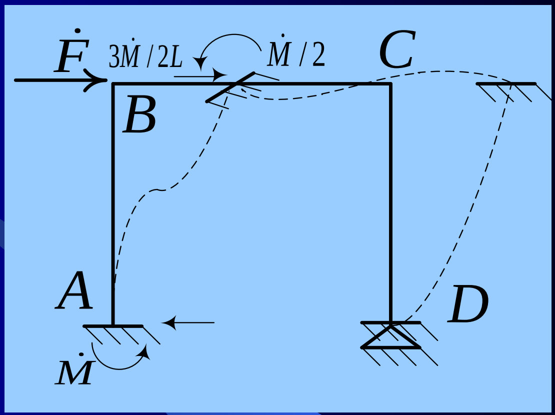


Indirect Method in the Nonlinear Incremental Analysis:

Continuation From Step 2:





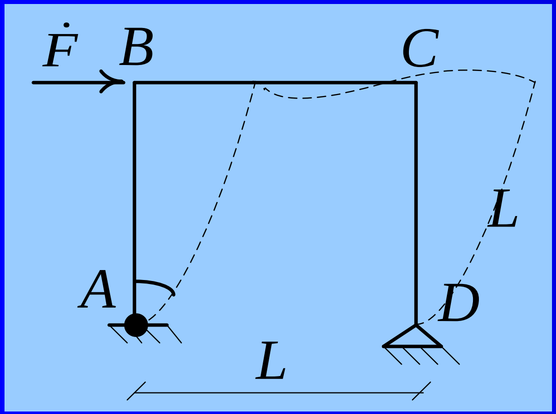


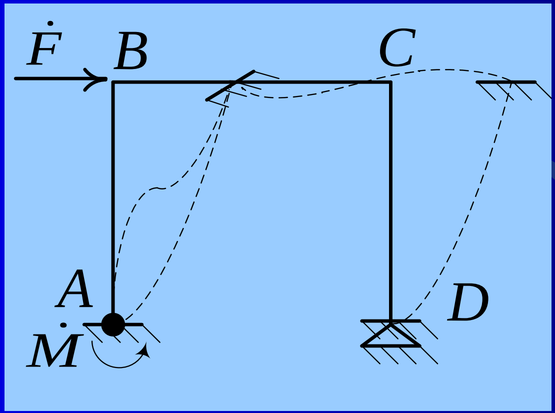
$$\begin{Bmatrix} \dot{Z}_{AB} \\ \dot{M}_{AB} \\ \dot{Z}_{BA} \\ \dot{M}_{BA} \end{Bmatrix} = \frac{EI}{l^3} \begin{bmatrix} 12 & -6l & -12 & -6l \\ -6l & 4l^2 & 6l & 2l^2 \\ -12 & 6l & 12 & 6l \\ -6l & 2l^2 & 6l & 4l^2 \end{bmatrix} \begin{Bmatrix} \dot{w}_A = 0 \\ \dot{\varphi}_A = -\dot{\alpha} \\ \vec{\dot{w}}_B = 0 \\ \dot{\varphi}_B = 0 \end{Bmatrix}$$

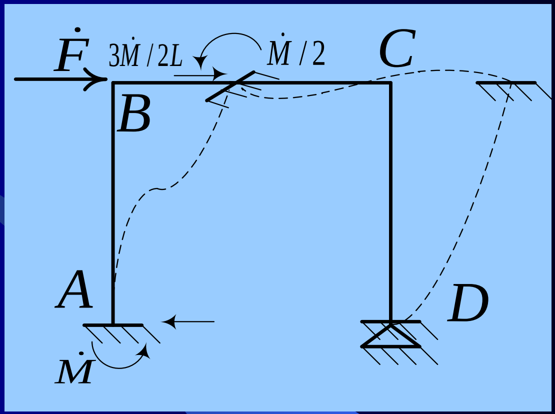
$$\begin{Bmatrix} \dot{Z}_{AB} \\ \dot{M}_{AB} \\ \dot{Z}_{BA} \\ \dot{M}_{BA} \end{Bmatrix} = \frac{-EI}{l^3} \dot{\alpha} \begin{Bmatrix} -6l \\ 4l^2 \\ 6l \\ 2l^2 \end{Bmatrix} = \begin{Bmatrix} -3\dot{M}/2l \\ \dot{M} \\ 3\dot{M}/2l \\ \dot{M}/2 \end{Bmatrix}$$

Indirect Method in the Nonlinear Incremental Analysis:

Continuation From Step 2, continue…:





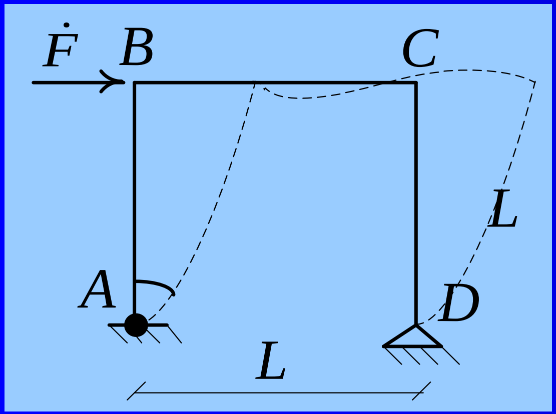


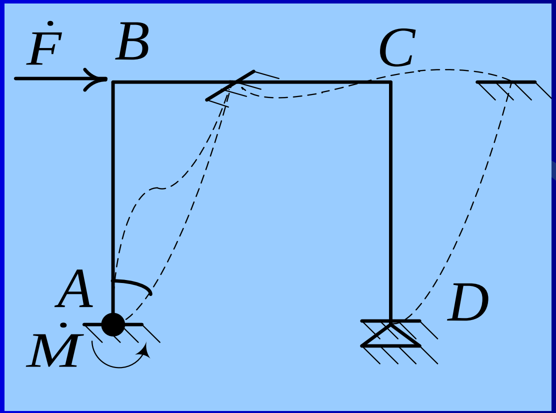
$$\frac{EI}{l^3}\begin{bmatrix} 15 & 6l & 3l \\ 6l & 8l^2 & 2l^2 \\ 3l & 2l^2 & 7l^2 \end{bmatrix}\begin{Bmatrix} \dot u \\ \dot\varphi_B \\ \dot\varphi_C \end{Bmatrix} = \begin{Bmatrix} \dot F \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 3\dot M/2l \\ \dot M/2 \\ 0 \end{Bmatrix}$$

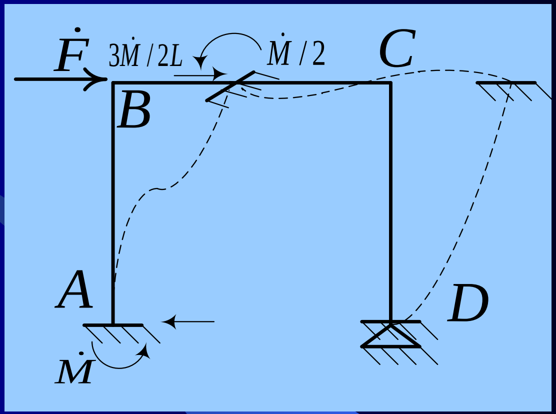
$$\begin{Bmatrix} \dot u \\ \dot\varphi_B \\ \dot\varphi_C \end{Bmatrix} = \begin{Bmatrix} 0.0985l \\ -0.0682 \\ -0.0227 \end{Bmatrix}\frac{\dot F l^2}{EI} + \begin{Bmatrix} 0.1136l \\ -0.0113 \\ -0.0455 \end{Bmatrix}\frac{\dot M l}{EI}$$

Indirect Method in the Nonlinear Incremental Analysis:

Continuation From Step 2, continue…:







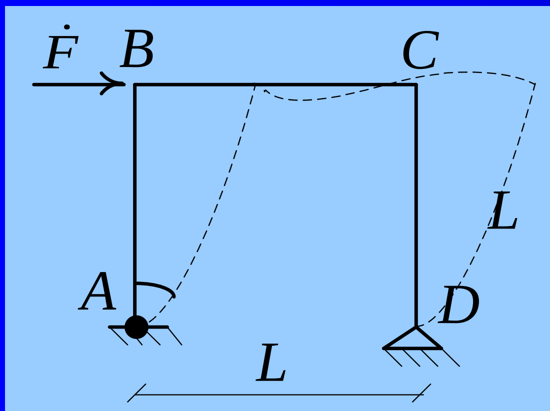
$$\begin{Bmatrix} \dot{M}_{AB} \\ \dot{M}_{BA} \\ \dot{M}_{BC} \\ \dot{M}_{CB} \\ \dot{M}_{CD} \end{Bmatrix} = \begin{Bmatrix} 0.4546 \\ 0.3182 \\ -0.3182 \\ -0.2272 \\ 0.2272 \end{Bmatrix} \dot{F}l + \begin{Bmatrix} 0.6590 \\ 0.6363 \\ -0.1363 \\ -0.2045 \\ 0.2045 \end{Bmatrix} \dot{M}$$

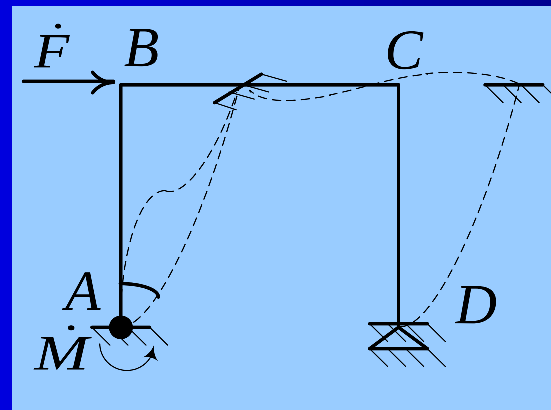
$$0.4546\dot{F}l + 0.6590\dot{M} = \dot{M}$$

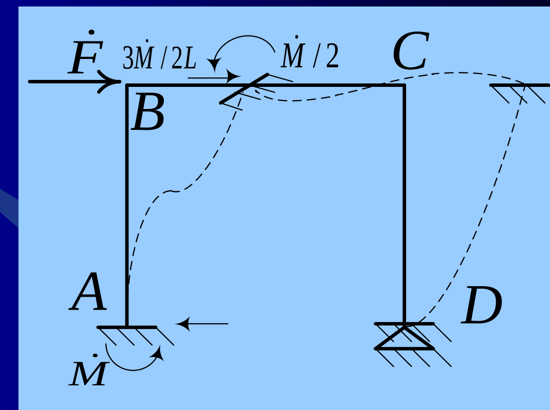
$$\dot{M} = 1.333\dot{F}l$$

Indirect Method in the Nonlinear Incremental Analysis:

Continuation From Step 2, continue…:





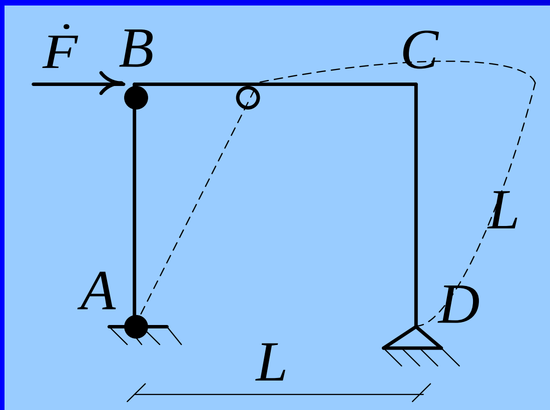


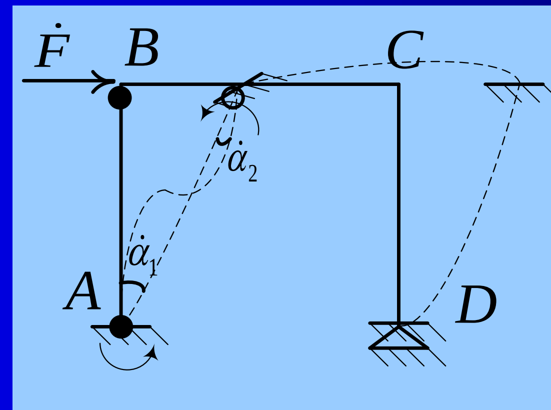
$$\begin{Bmatrix} \dot{M}_{AB} \\ \dot{M}_{BA} \\ \dot{M}_{BC} \\ \dot{M}_{CB} \\ \dot{M}_{CD} \end{Bmatrix} = \begin{Bmatrix} 1.3\cancel{3}3 \\ 1.1\cancel{6}6 \\ -0.5000 \\ -0.5000 \\ 0.5000 \end{Bmatrix} \dot{F}l$$

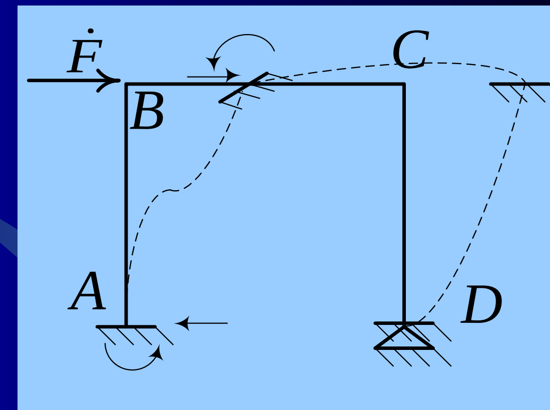
$$\begin{Bmatrix} \dot{M}_{AB} \\ \dot{M}_{BA} \\ \dot{M}_{BC} \\ \dot{M}_{CB} \\ \dot{M}_{CD} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.5000 \\ -0.5000 \\ -0.5000 \\ 0.5000 \end{Bmatrix} \dot{F}l$$

Indirect Method in the Nonlinear Incremental Analysis:

Continuation From Step 3:





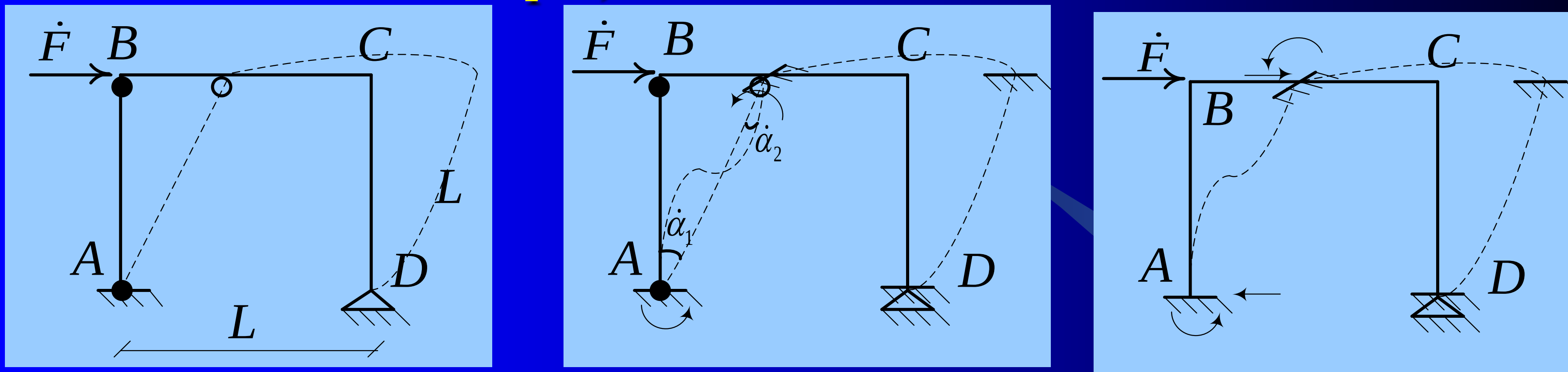


$$\begin{Bmatrix} \dot{Z}_{AB} \\ \dot{M}_{AB} \\ \dot{Z}_{BA} \\ \dot{M}_{BA} \end{Bmatrix} = \frac{EI}{l^3} \begin{bmatrix} 12 & -6l & -12 & -6l \\ -6l & 4l^2 & 6l & 2l^2 \\ -12 & 6l & 12 & 6l \\ -6l & 2l^2 & 6l & 4l^2 \end{bmatrix} \begin{Bmatrix} \dot{w}_A = 0 \\ \dot{\varphi}_A = -\dot{\alpha}_1 \\ \vec{\dot{w}}_B = 0 \\ \dot{\varphi}_B = -\dot{\alpha}_2 \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{Z}_{AB} \\ \dot{M}_{AB} \\ \dot{Z}_{BA} \\ \dot{M}_{BA} \end{Bmatrix} = \frac{-EI}{l^3}\dot{\alpha}_1 \begin{Bmatrix} -6l \\ 4l^2 \\ 6l \\ 2l^2 \end{Bmatrix} + \frac{-EI}{l^3}\dot{\alpha}_2 \begin{Bmatrix} -6l \\ 2l^2 \\ 6l \\ 4l^2 \end{Bmatrix} = \begin{Bmatrix} -3\dot{M}_1/2l \\ \dot{M}_1 \\ 3\dot{M}_1/2l \\ \dot{M}_1/2 \end{Bmatrix} + \begin{Bmatrix} -3\dot{M}_2/2l \\ \dot{M}_2/2 \\ 3\dot{M}_2/2l \\ \dot{M}_2 \end{Bmatrix}$$

Indirect Method in the Nonlinear Incremental Analysis:

Continuation From Step 3, continue…:

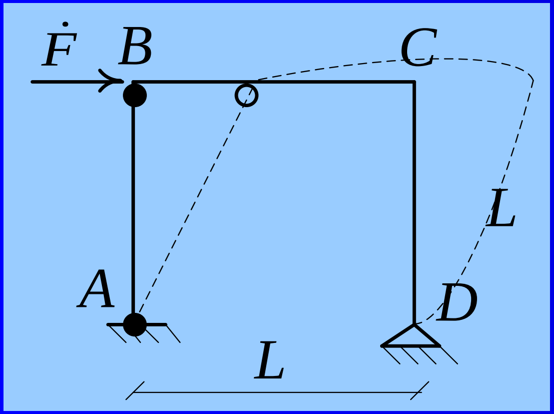


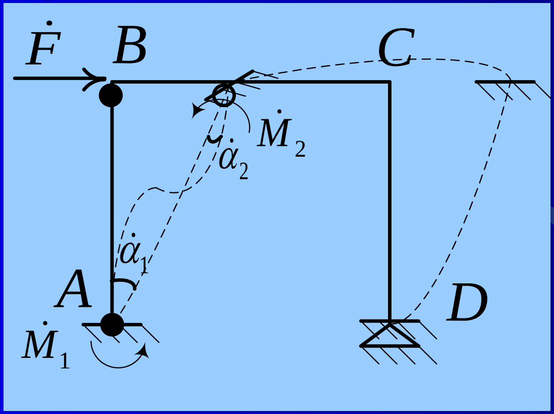
$$\frac{EI}{l^3}\begin{bmatrix} 15 & 6l & 3l \\ 6l & 8l^2 & 2l^2 \\ 3l & 2l^2 & 7l^2 \end{bmatrix}\begin{Bmatrix} \dot{u} \\ \dot{\varphi}_B \\ \dot{\varphi}_C \end{Bmatrix} = \begin{Bmatrix} \dot{F} \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 3\dot{M}_1/2l \\ \dot{M}_1/2 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 3\dot{M}_2/2l \\ \dot{M}_2 \\ 0 \end{Bmatrix}$$

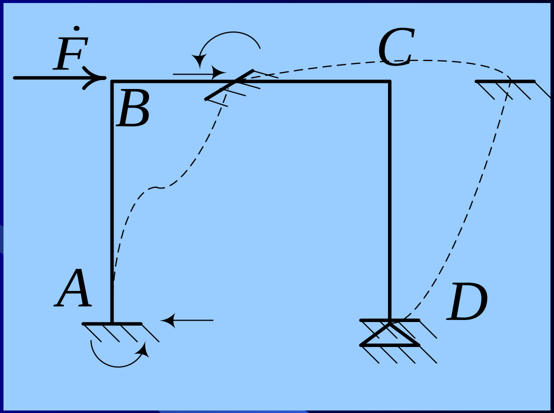
$$\begin{Bmatrix} \dot{u} \\ \dot{\varphi}_B \\ \dot{\varphi}_C \end{Bmatrix} = \begin{Bmatrix} 0.0985l \\ -0.0682 \\ -0.0227 \end{Bmatrix}\frac{\dot{F}l^2}{EI} + \begin{Bmatrix} 0.1136l \\ -0.0113 \\ -0.0455 \end{Bmatrix}\frac{\dot{M}_1 l}{EI} + \begin{Bmatrix} 0.0795l \\ 0.0795 \\ -0.0568 \end{Bmatrix}\frac{\dot{M}_2 l}{EI}$$

Indirect Method in the Nonlinear Incremental Analysis:

Continuation From Step 3, continue…:







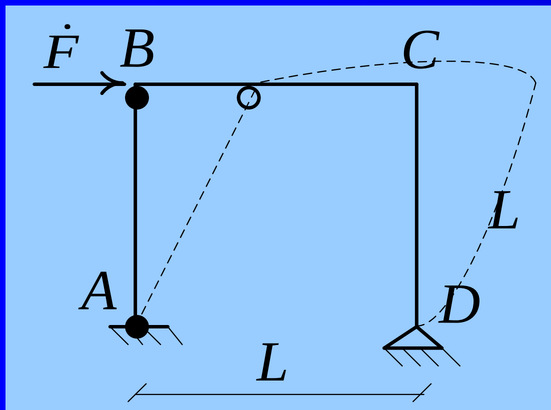
$$\begin{Bmatrix} \dot{M}_{AB} \\ \dot{M}_{BA} \\ \dot{M}_{BC} \\ \dot{M}_{CB} \\ \dot{M}_{CD} \end{Bmatrix} = \begin{Bmatrix} 0.4546 \\ 0.3182 \\ -0.3182 \\ -0.2272 \\ 0.2272 \end{Bmatrix} \dot{F}l + \begin{Bmatrix} 0.6590 \\ 0.6363 \\ -0.1363 \\ -0.2045 \\ 0.2045 \end{Bmatrix} \dot{M}_1 + \begin{Bmatrix} 0.6363 \\ 0.7950 \\ 0.2044 \\ -0.0682 \\ 0.0682 \end{Bmatrix} \dot{M}_2$$

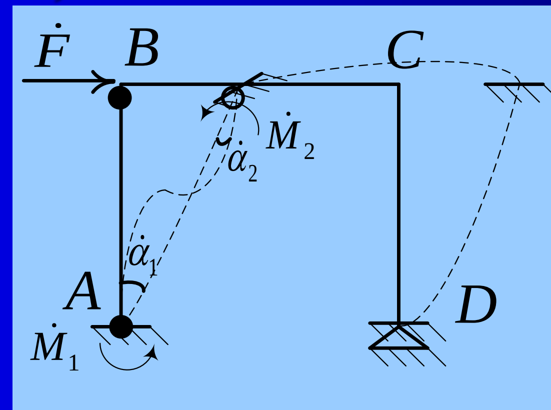
$$\begin{cases} 0.4546\dot{F}l + 0.6590\dot{M}_1 + 0.6363\dot{M}_2 = \dot{M}_1 + \dot{M}_2 / 2 \\ 0.3182\dot{F}l + 0.6363\dot{M}_1 + 0.7950\dot{M}_2 = \dot{M}_2 + \dot{M}_1 / 2 \end{cases}$$

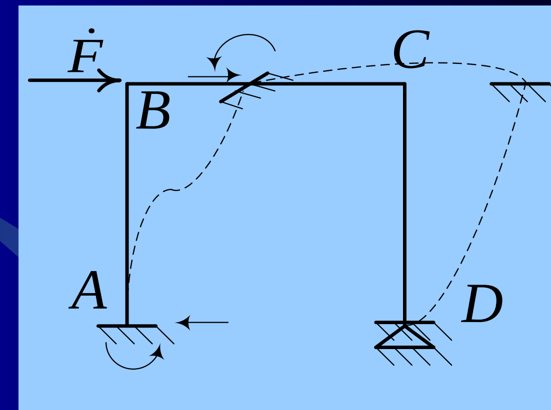
$$\begin{cases} \dot{M}_1 = 2.661\dot{F}l \\ \dot{M}_2 = 3.321\dot{F}l \end{cases}$$

Indirect Method in the Nonlinear Incremental Analysis:

Continuation From Step 3, continue…:







$$\begin{Bmatrix} \dot{M}_{AB} \\ \dot{M}_{BA} \\ \dot{M}_{BC} \\ \dot{M}_{CB} \\ \dot{M}_{CD} \end{Bmatrix} = \begin{Bmatrix} 4.\cancel{3}21 \\ 4.6\cancel{5}2 \\ 0 \\ -1 \\ 1 \end{Bmatrix} \dot{F}l$$

$$\begin{Bmatrix} \dot{M}_{AB} \\ \dot{M}_{BA} \\ \dot{M}_{BC} \\ \dot{M}_{CB} \\ \dot{M}_{CD} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{Bmatrix} \dot{F}l$$