Plastic Analysis and Design of Structures

Part 1-2

S. Erfani

Moment Curvature and Force Elongation Relations in Axial-Flexural Members:

-Bending Usual Assumption (Bernoulli Assumption):

"The Plane Cross Sections Remain Plain and Normal to The Deflected Middle Axis of The Beam".

$z : The\ Fiber\ Depth\ Coordinate\ Measured\ From\ The\ Middle\ Axis$

$\kappa : The\ Middle\ Axis\ Curvature$

$\bar{\varepsilon} : Strain\ of\ The\ Middle\ Axis$

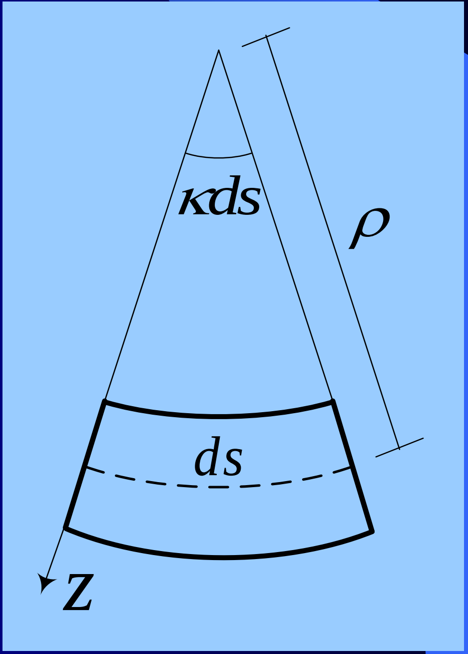
$\varepsilon = z\kappa + \bar{\varepsilon}$

$\rho : Radius\ of\ Curvature$

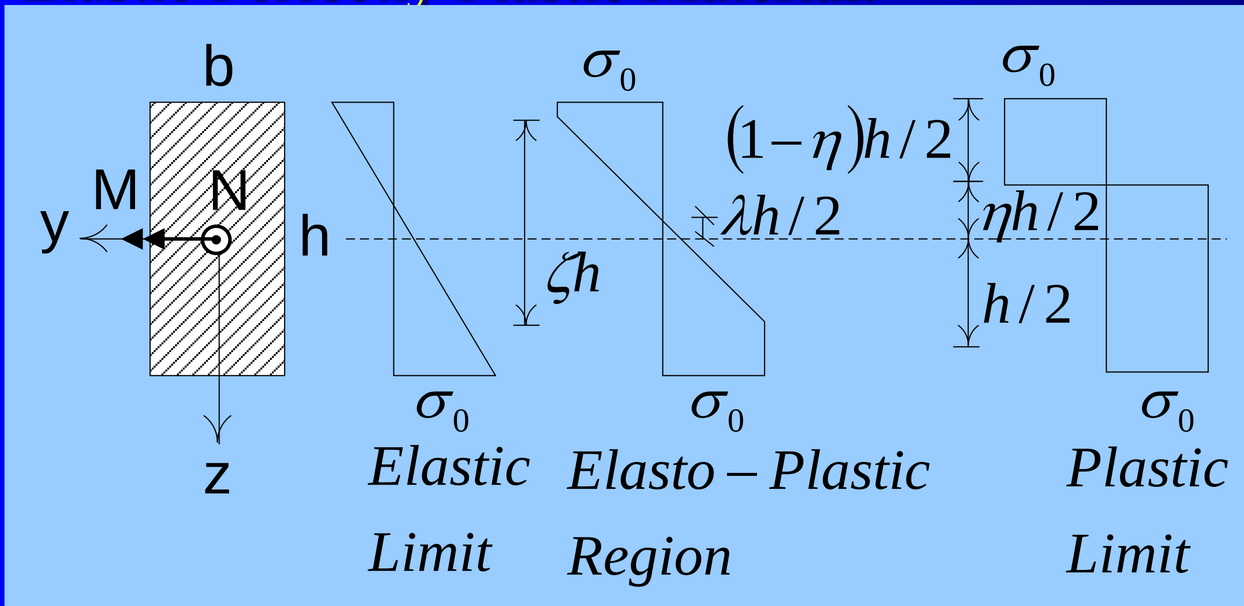
$$\kappa = \frac{1}{\rho}$$

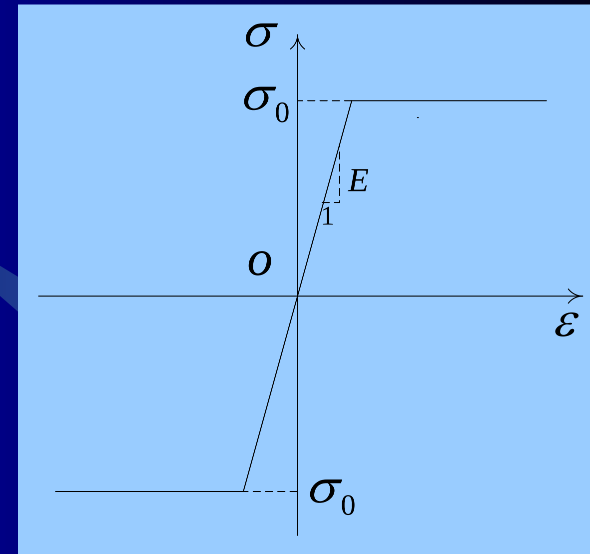
$$M = \int_A \sigma(z) z dA$$

$$N = \int_A \sigma(z) dA$$



Example: Rectangular Cross Section With Elastic Perfectly Plastic Material:





Elastic Region:

$$M = \int_A z\sigma(z)dA = \int_A zE\varepsilon(z)dA$$

$$M = \int_A zE\left(z\kappa + \bar{\varepsilon}\right)dA = E\kappa\int_A z^2 dA + E\bar{\varepsilon}\int_A \cancel{z}dA$$

$$N = \int_A \sigma(z)dA = \int_A E\varepsilon(z)dA$$

$$N = \int_A E\left(z\kappa + \bar{\varepsilon}\right)dA = E\kappa\int_A \cancel{z}dA + E\bar{\varepsilon}\int_A dA$$

$$\int_A z^2 dA = I$$

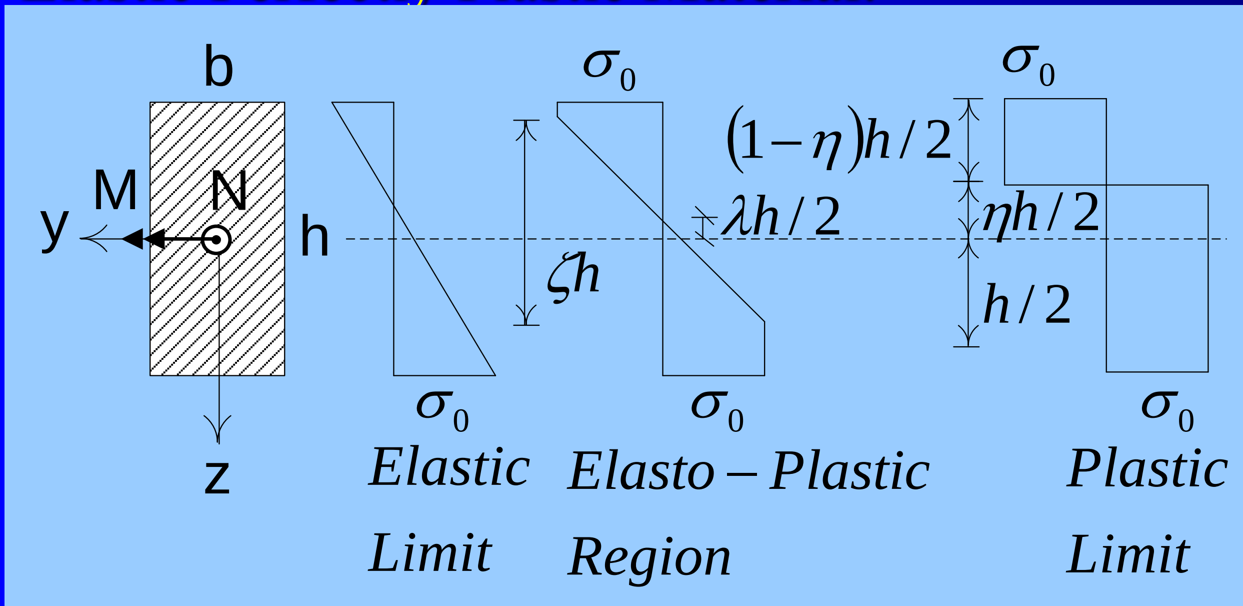
$$\int_A z dA = 0$$

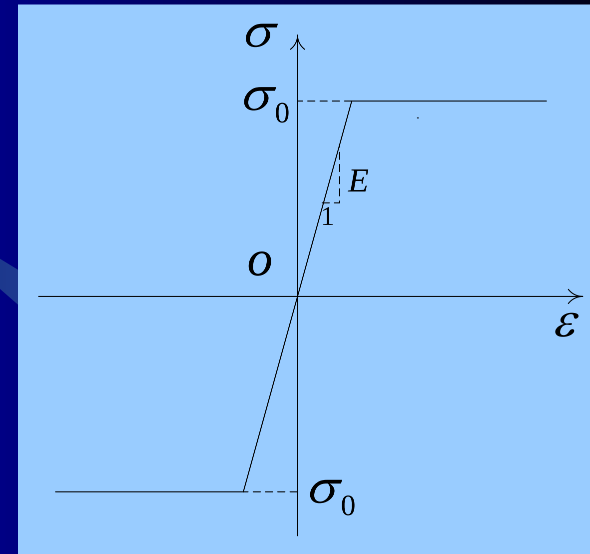
$$\int_A dA = A$$

$$M = EI\kappa$$

$$N = EA\bar{\varepsilon}$$

Example: Rectangular Cross Section With Elastic Perfectly Plastic Material:





Elastic Limit:

$$\sigma(z = h/2) = \sigma_0$$

$$E\varepsilon(z = h/2) = \sigma_0$$

$$E\left(\frac{h}{2}\kappa + \bar{\varepsilon}\right) = \sigma_0$$

$$E\left(\frac{h}{2}\frac{M}{EI} + \frac{N}{EA}\right) = \sigma_0$$

$$\frac{h}{2}\frac{M}{I} + \frac{N}{A} = \sigma_0$$

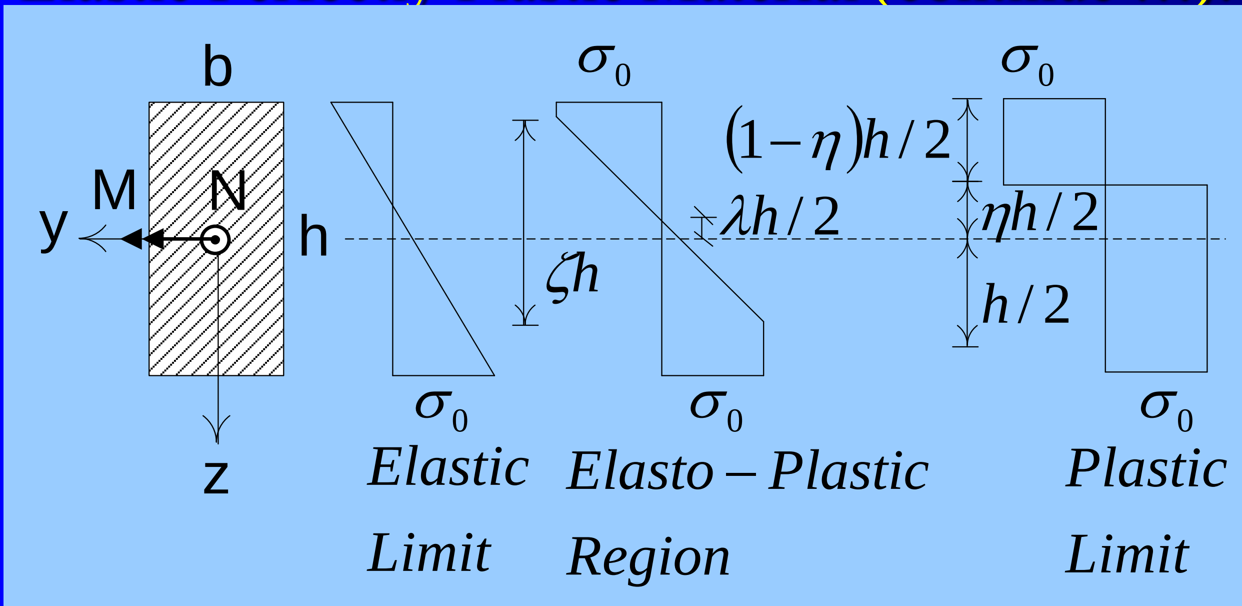
$$\frac{M}{\sigma_0 W_{el}} + \frac{N}{\sigma_0 A} = 1$$

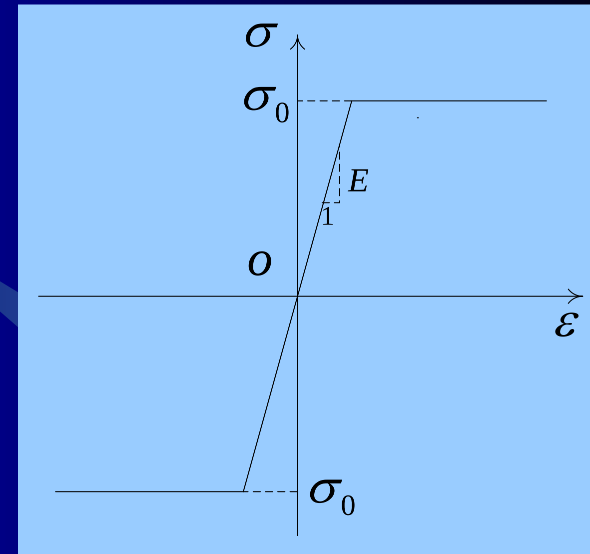
$$\sigma_0 W_{el} = M_{el}$$

$$\sigma_0 A = N_0$$

$$\frac{M}{M_{el}} + \frac{N}{N_0} = 1$$

Example: Rectangular Cross Section With Elastic Perfectly Plastic Material (continue …):





Plastic Limit:

$$M = \int_A z\sigma(z)\,dA$$

$$M = \sigma_0 b\left[(1-\eta)\frac{h}{2}(1+\eta)\frac{h}{4} + (1+\eta)\frac{h}{2}(1-\eta)\frac{h}{4}\right]$$

$$M = \frac{\sigma_0 bh^2}{4}\left(1-\eta^2\right)$$

$$\frac{bh^2}{4} = W_0$$

$$\sigma_0 W_0 = M_0$$

$$M = M_0\left(1-\eta^2\right)$$

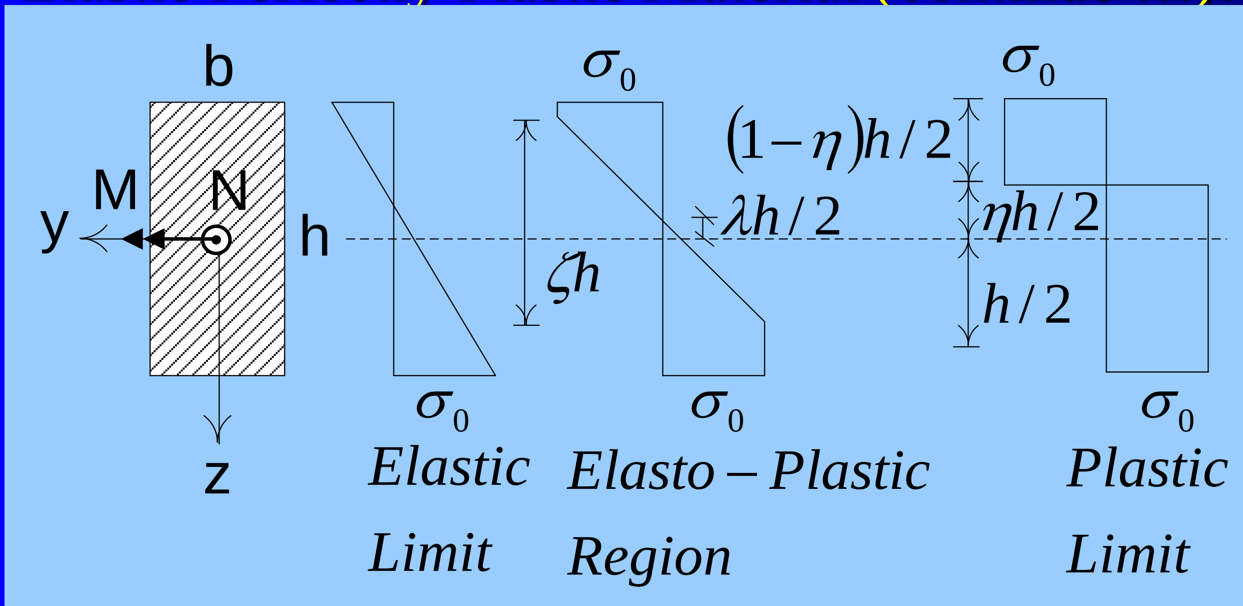
$$N = \int_A \sigma(z)\,dA$$

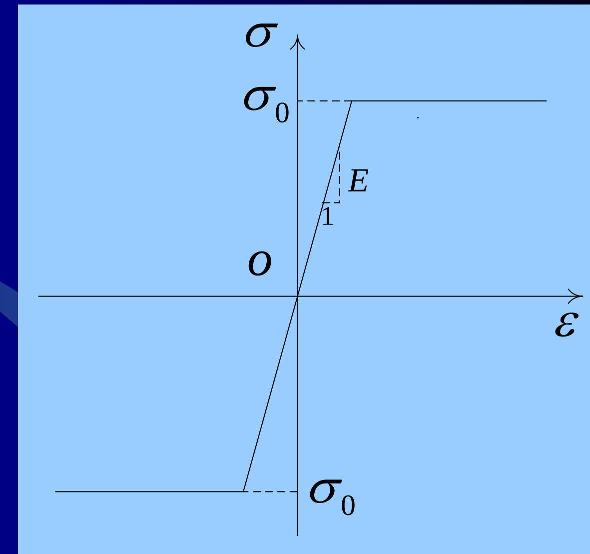
$$N = \sigma_0 b\left[(1+\eta)\frac{h}{2} - (1-\eta)\frac{h}{2}\right]$$

$$N = \sigma_0 bh\eta$$

$$N = N_0\eta$$

Example: Rectangular Cross Section With Elastic Perfectly Plastic Material (continue …):





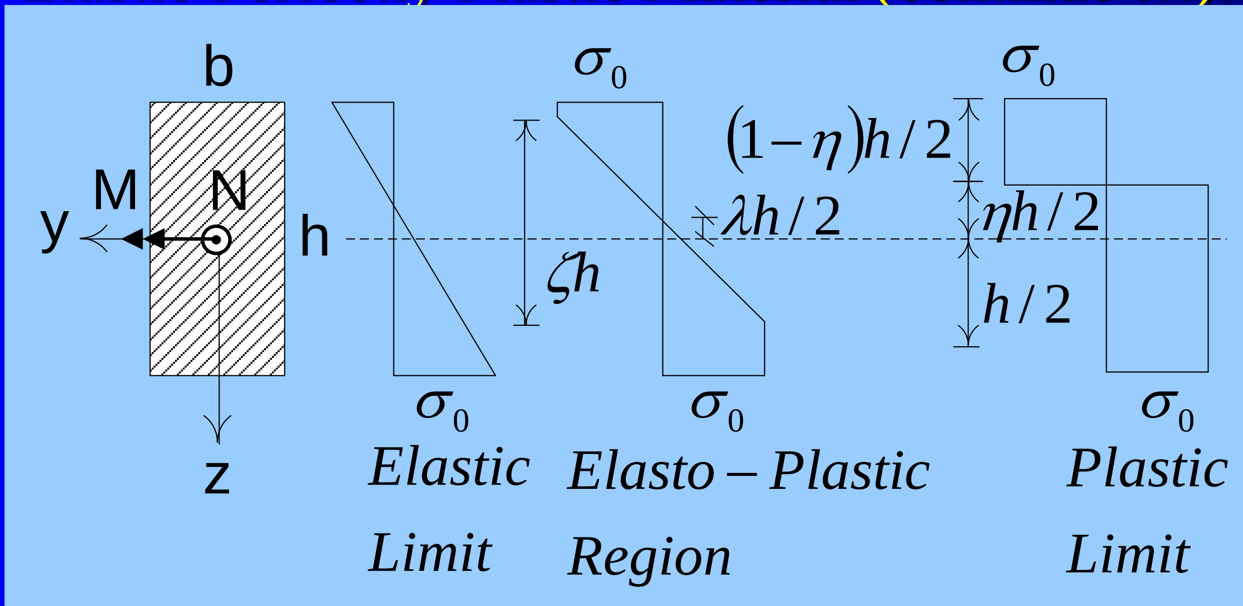
Plastic Limit (continue…):

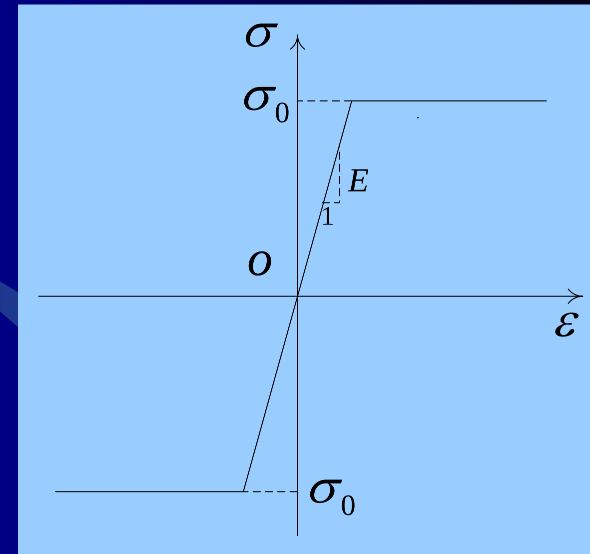
$$\begin{cases} M = M_0\left(1-\eta^2\right) \\ N = N_0\eta \end{cases}$$

$$\begin{cases} M/M_0 = 1-\eta^2 \\ N/N_0 = \eta \end{cases}$$

$$\frac{M}{M_0} + \left(\frac{N}{N_0}\right)^2 = 1$$

Example: Rectangular Cross Section With Elastic Perfectly Plastic Material (continue …):



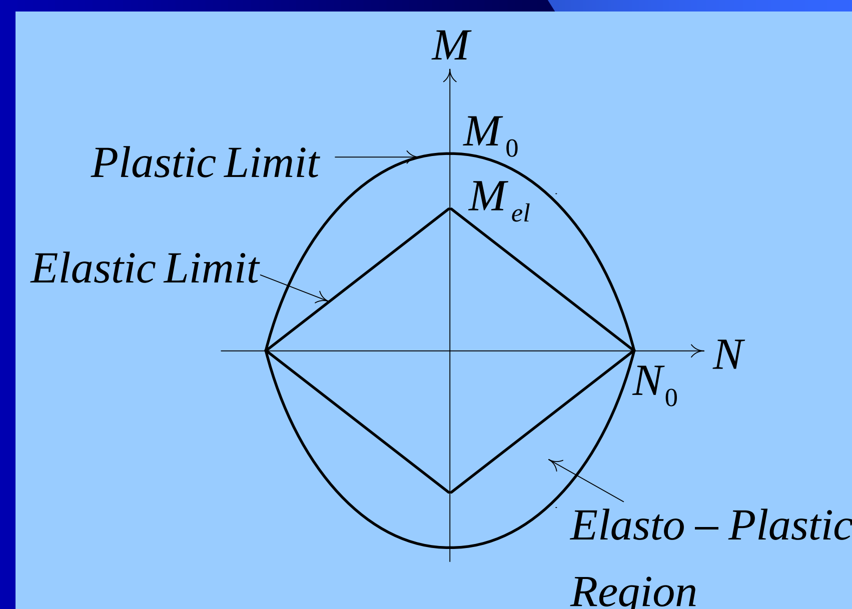


Elastic Limit :

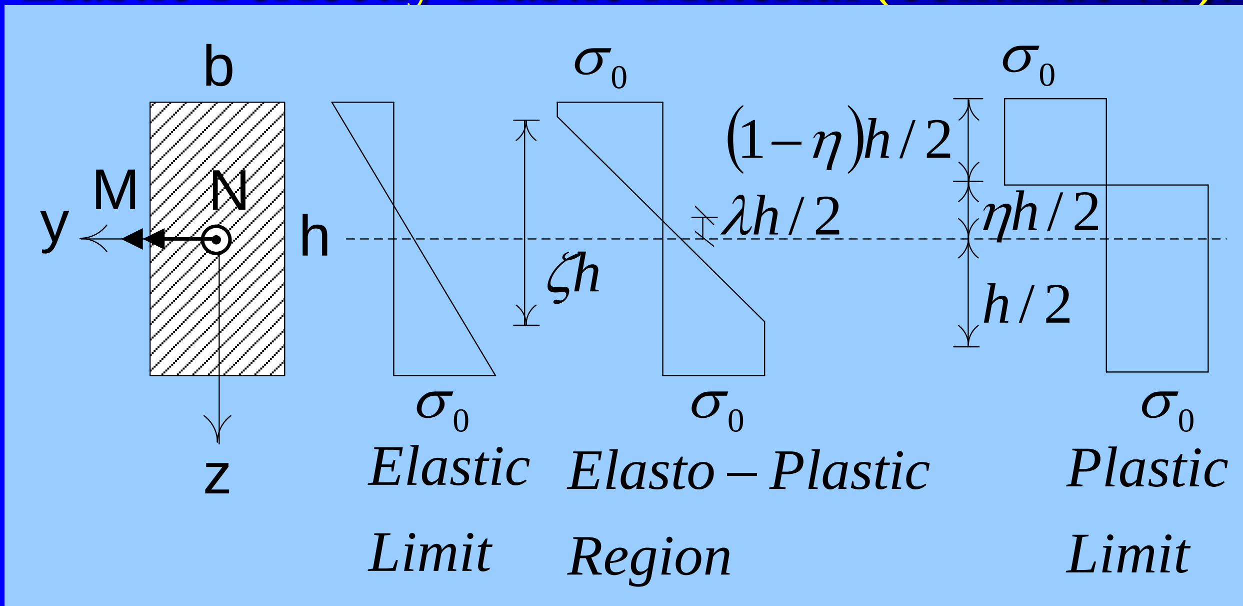
$$\frac{|M|}{M_{el}} + \frac{|N|}{N_0} = 1$$

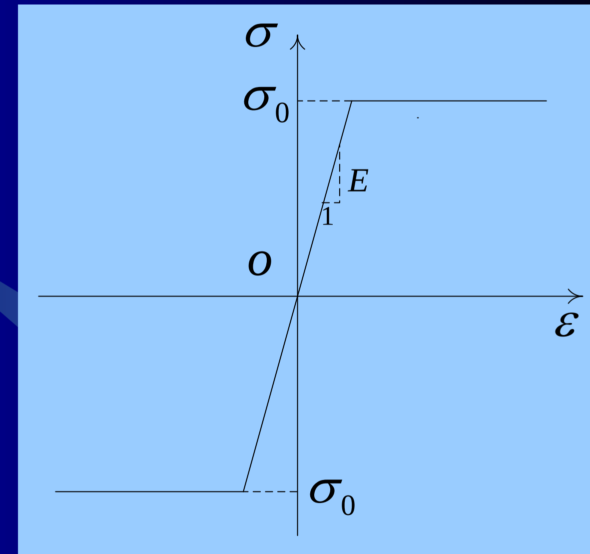
Plastic Limit :

$$\frac{|M|}{M_0} + \left(\frac{N}{N_0}\right)^2 = 1$$



Example: Rectangular Cross Section With Elastic Perfectly Plastic Material (continue …):





Elasto-Plastic Region:

$$\begin{cases} M = M_0\left(1-\lambda^2-\dfrac{\zeta^2}{3}\right) \\ N = N_0\lambda \end{cases}$$

$$\begin{cases} \dfrac{M}{M_0} = 1-\lambda^2-\dfrac{\zeta^2}{3} \\ \dfrac{N}{N_0} = \lambda \end{cases}$$

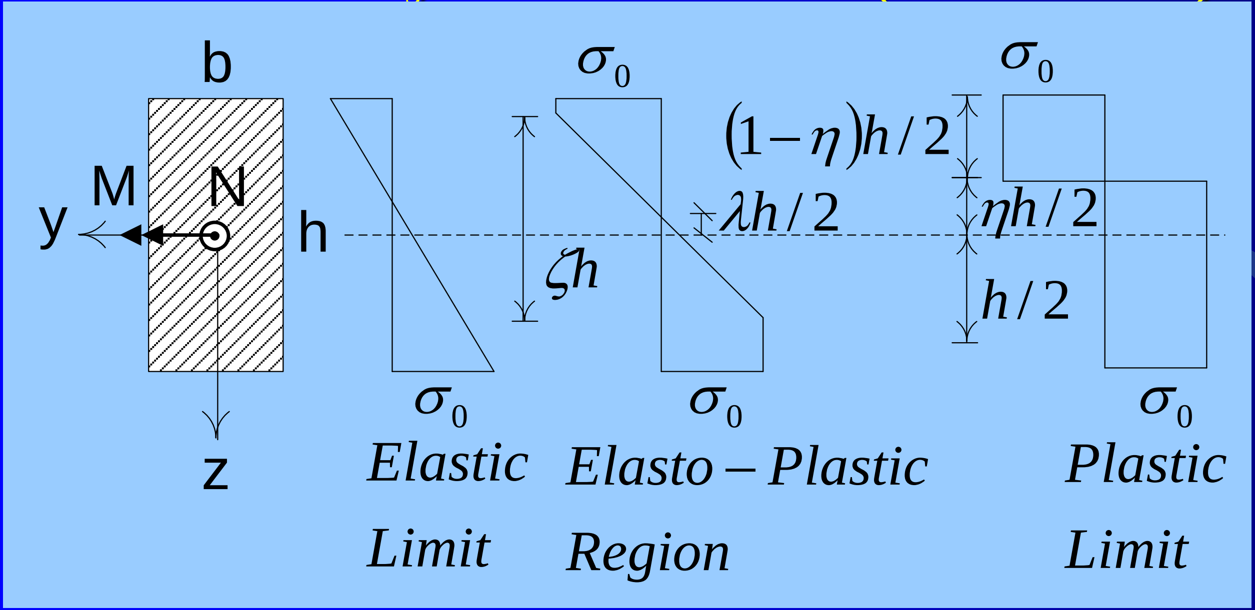
$$\frac{M}{M_0}+\left(\frac{N}{N_0}\right)^2 = 1-\frac{\zeta^2}{3}$$

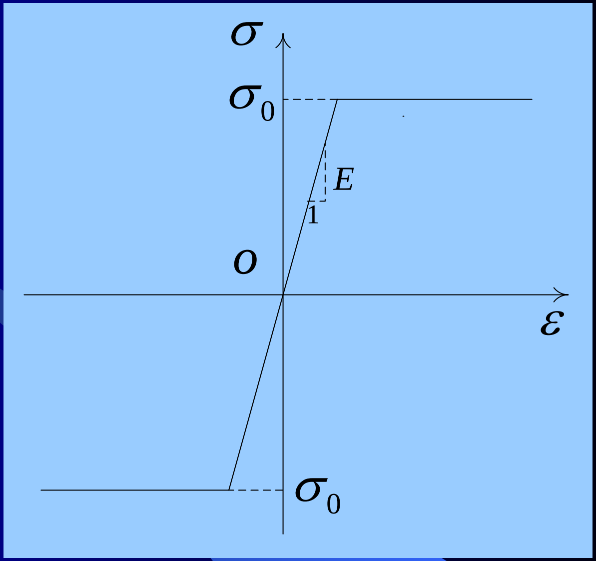
$$2\varepsilon_0 = \zeta h\kappa$$

$$\zeta = \frac{2\varepsilon_0}{h\kappa}$$

$$\frac{M}{M_0}+\left(\frac{N}{N_0}\right)^2 = 1-\frac{4\varepsilon_0^2}{3h^2\kappa^2}$$

Example: Rectangular Cross Section With Elastic Perfectly Plastic Material (continue …):





Elasto-Plastic Region (continue…):

$$\varepsilon_0 \frac{\lambda}{\zeta} = \bar{\varepsilon}$$

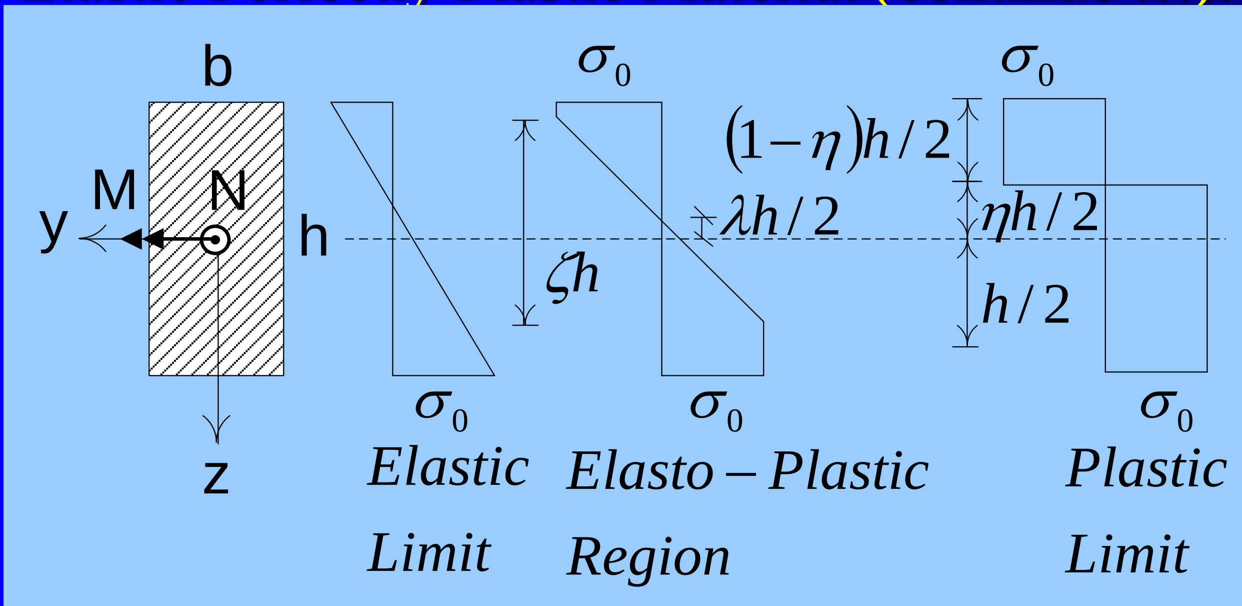
$$\frac{\varepsilon_0}{\bar{\varepsilon}} \lambda = \zeta$$

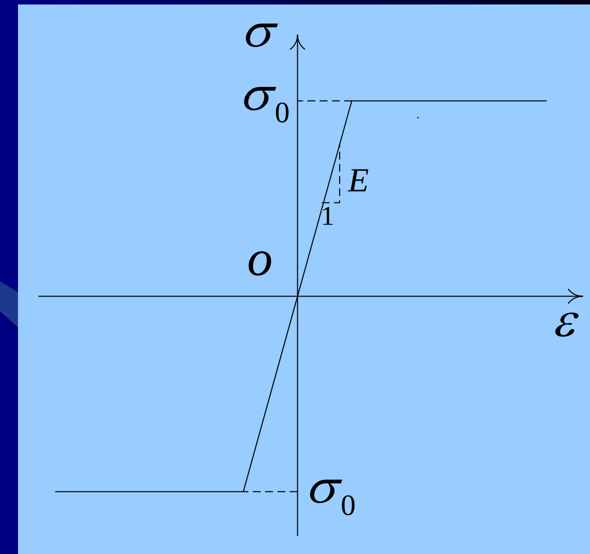
$$\frac{M}{M_0} + \left(\frac{N}{N_0}\right)^2 = 1 - \frac{\zeta^2}{3}$$

$$\frac{M}{M_0} + \left(\frac{N}{N_0}\right)^2 = 1 - \frac{\varepsilon_0^2 \lambda^2}{3\bar{\varepsilon}^2}$$

$$\frac{M}{M_0} + \left(\frac{N}{N_0}\right)^2 = 1 - \frac{\varepsilon_0^2}{3\bar{\varepsilon}^2}\left(\frac{N}{N_0}\right)^2$$

Example: Rectangular Cross Section With Elastic Perfectly Plastic Material (continue …):





Elasto-Plastic Region (continue…):

$$\kappa(M,N)=\frac{2\varepsilon_0}{h\sqrt{3\left(1-\frac{M}{M_0}-\left(\frac{N}{N_0}\right)^2\right)}}$$

$$\bar{\varepsilon}(M,N)=\frac{\varepsilon_0}{\sqrt{3\left(1-\frac{M}{M_0}-\left(\frac{N}{N_0}\right)^2\right)}}\frac{N}{N_0}$$

$$\bar{\varepsilon}(M,N)=\frac{h}{2}\frac{N}{N_0}\kappa(M,N)$$

Plastic Limit: $\kappa,\bar{\varepsilon}\rightarrow\infty$

Expanded Plastic zone:

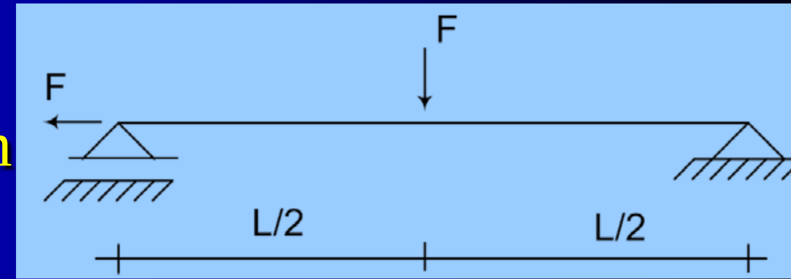
Example) Rectangular Cross Section Beam With Elastic Perfectly Plastic Material:

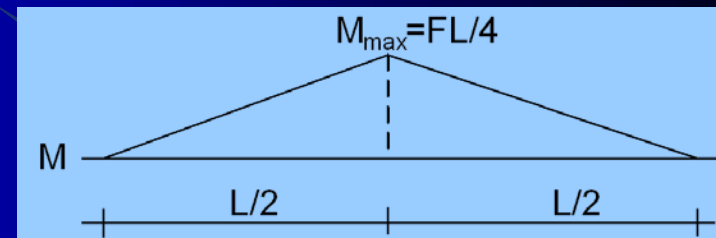
$$M(x) = \frac{1}{2}Fx, \quad x \in \left[0, \frac{L}{2}\right]$$

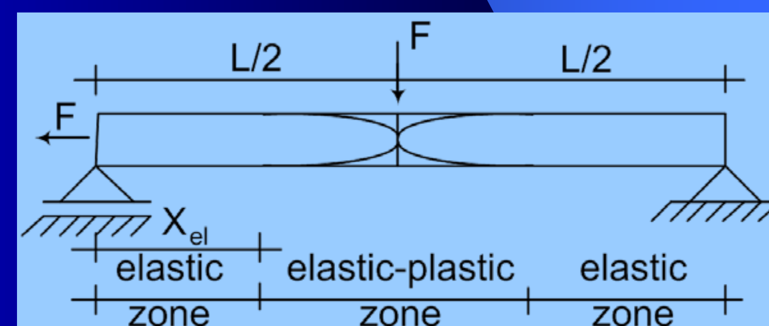
$$x = \frac{L}{2} \rightarrow M_{\max} = \frac{FL}{4}$$

$$M(x) = \frac{2M_{\max}}{L}x = \frac{F}{2}x, \quad x \in \left[0, \frac{L}{2}\right]$$

$$N(x) = F, \quad x \in \left[0, \frac{L}{2}\right]$$

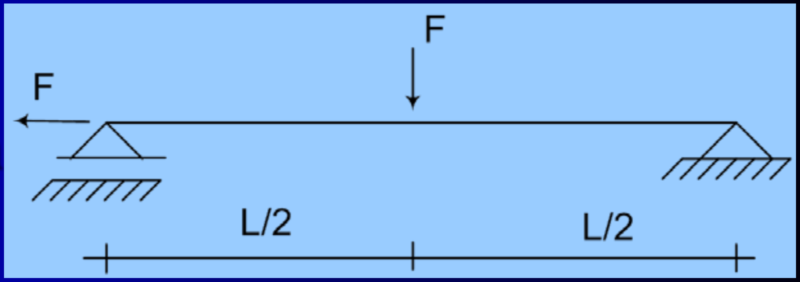


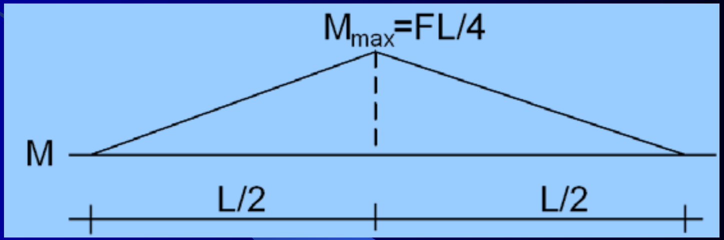


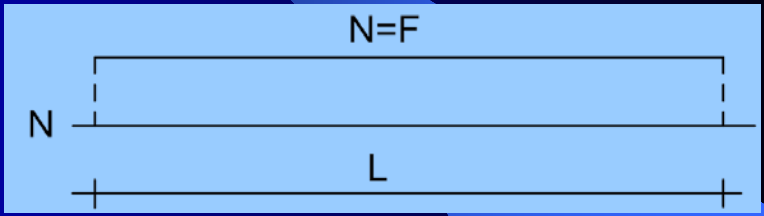


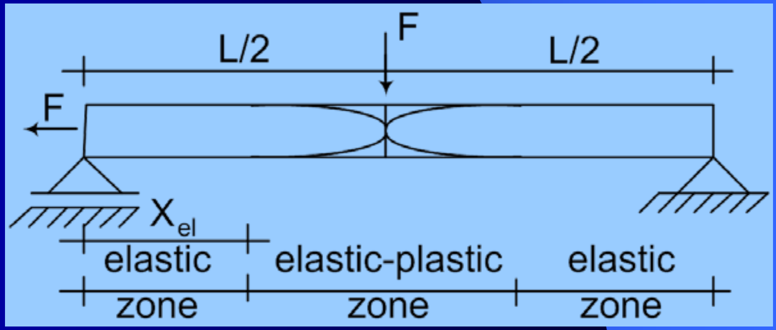
Expanded Plastic zone:

Example) Rectangular Cross Section Beam With Elastic Perfectly Plastic Material:









Elastic Limit :

$$\frac{|M_{\max}|}{M_{el}} + \frac{|N|}{N_0} = 1$$

$$\frac{\frac{F_{el}L}{4}}{\sigma_0 \frac{bh^2}{6}} + \frac{F_{el}}{\sigma_0 bh} = 1$$

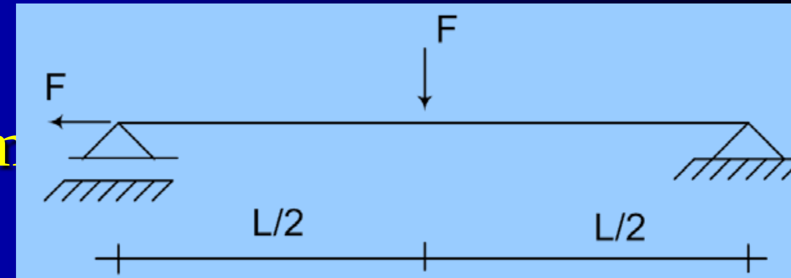
$$F_{el} = \frac{\sigma_0 bh}{1 + \frac{3L}{2h}}$$

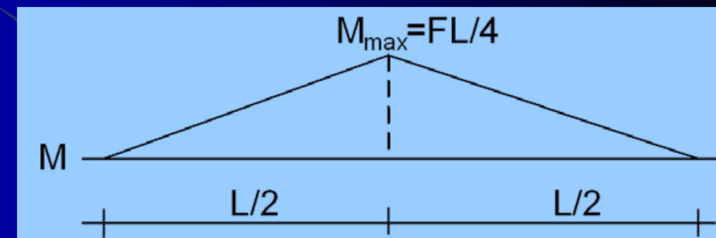
$$\frac{L}{2h} = \gamma$$

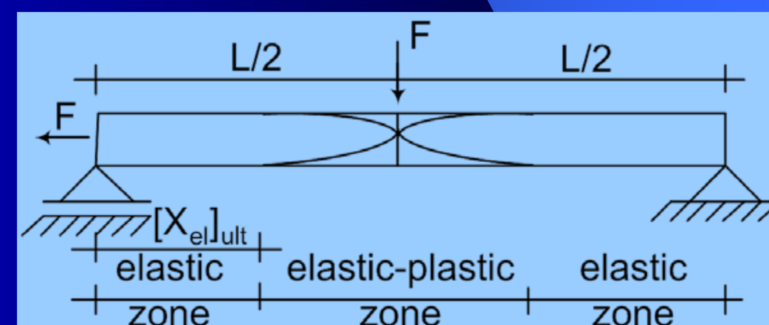
$$F_{el} = \frac{N_0}{1 + 3\gamma}$$

Expanded Plastic zone:

Example) Rectangular Cross Section Beam With Elastic Perfectly Plastic Material:







Plastic Limit :

$$\frac{|M|}{M_0}+\left(\frac{N}{N_0}\right)^2=1$$

$$\frac{\frac{F_0L}{4}}{\sigma_0\frac{bh^2}{4}}+\left(\frac{F_0}{\sigma_0 bh}\right)^2=1$$

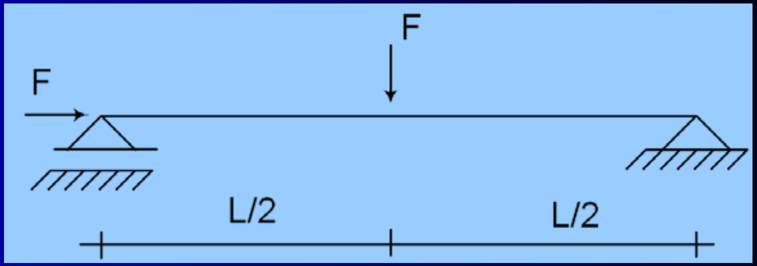
$$\left(\frac{F_0}{\sigma_0 bh}\right)^2+\frac{L}{h}\left(\frac{F_0}{\sigma_0 bh}\right)-1=0$$

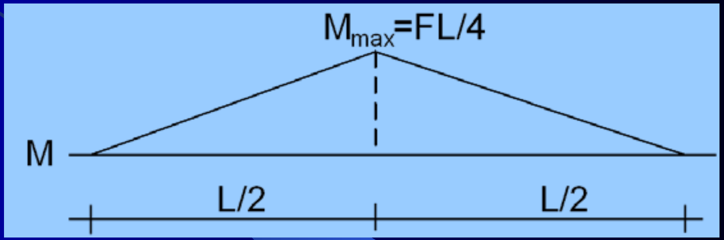
$$F_0=\sigma_0 bh\left(\frac{-\frac{L}{h}+\sqrt{\frac{L^2}{h^2}+4}}{2}\right)$$

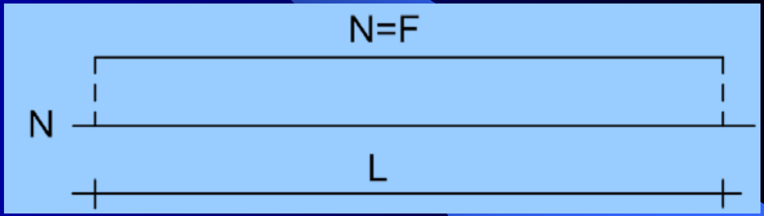
$$F_0=N_0\left(-\gamma+\sqrt{\gamma^2+1}\right)$$

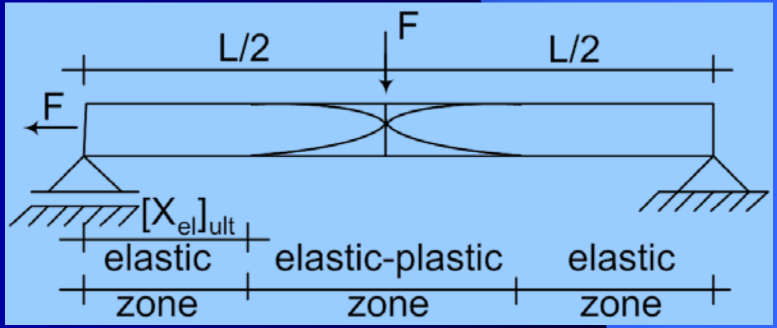
Expanded Plastic zone:

Example) Rectangular Cross Section Beam With Elastic Perfectly Plastic Material:









$$F_{el} \le F \le F_0 :$$

$$x = x_{el} \rightarrow \frac{|M(x)|}{M_{el}} + \frac{|N|}{N_0} = 1$$

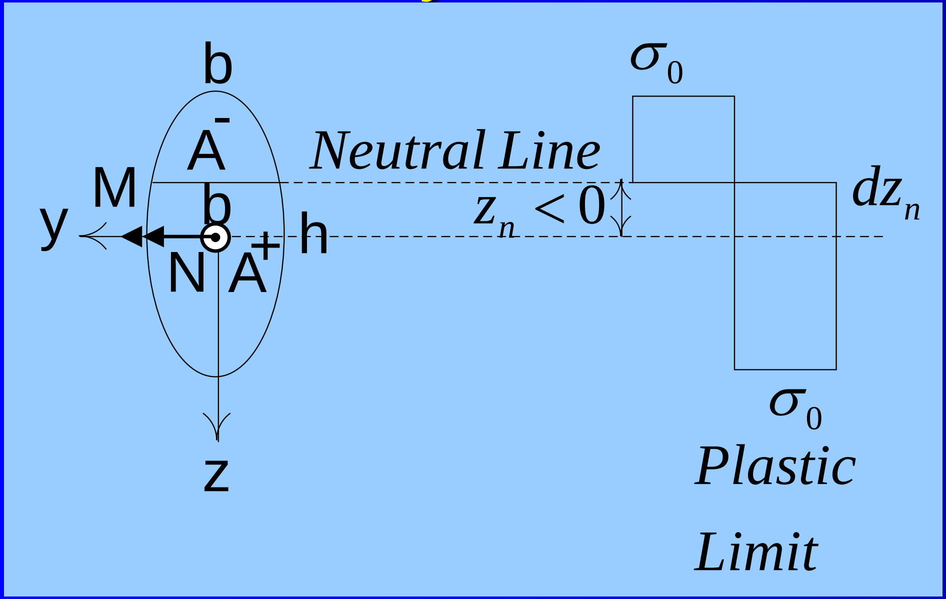
$$\frac{\dfrac{Fx_{el}}{2}}{\sigma_0 \dfrac{bh^2}{6}} + \frac{F}{\sigma_0 bh} = 1$$

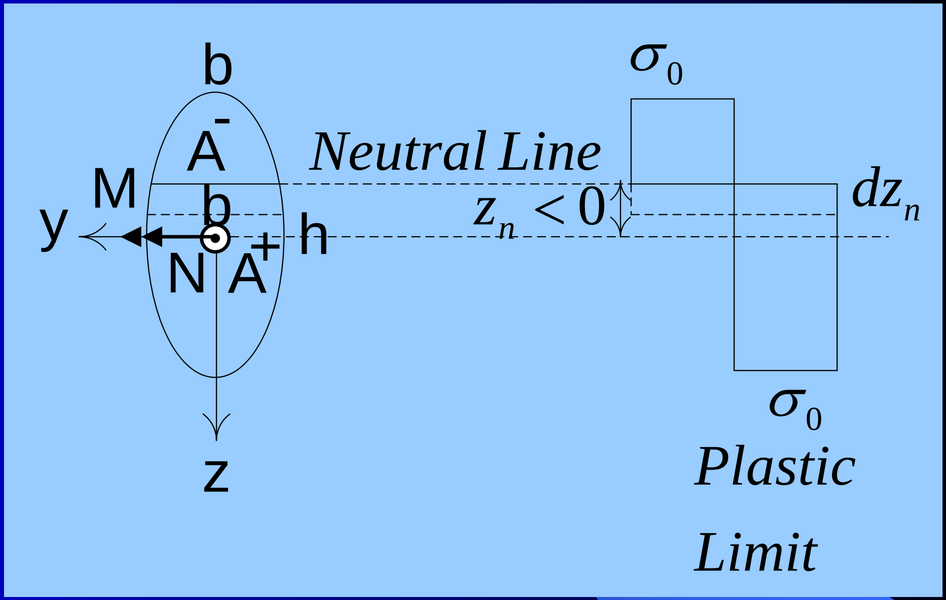
$$x_{el} = L\frac{2\gamma}{3}\left(\frac{N_0}{F} - 1\right)$$

$$[x_{el}]_{ult} = L\frac{2\gamma}{3}\left(\frac{1}{\sqrt{1+\gamma^2}-1} - 1\right)$$

$$\frac{F_0}{F_{el}} = (1+3\gamma)\left(-\gamma + \sqrt{\gamma^2+1}\right)$$

Plastic Limit of General Cross Section With Elastic Perfectly Plastic Material:





$$N = \sigma_0 \left(A^+ - A^- \right)$$

$$M = \sigma_0 \left(S^+ - S^- \right)$$

$$S^+ = \int_{A^+} z dA$$

$$S^- = \int_{A^-} z dA$$

$$\begin{cases} dA^+ = -b_n dz_n \\ dA^- = +b_n dz_n \end{cases}$$

$$dN = -2\sigma_0 b_n dz_n$$

$$\frac{dM}{dN} = z_n$$

$$\begin{cases} dS^+ = -b_n z_n dz_n \\ dS^- = +b_n z_n dz_n \end{cases}$$

$$dM = -2\sigma_0 b_n z_n dz_n$$

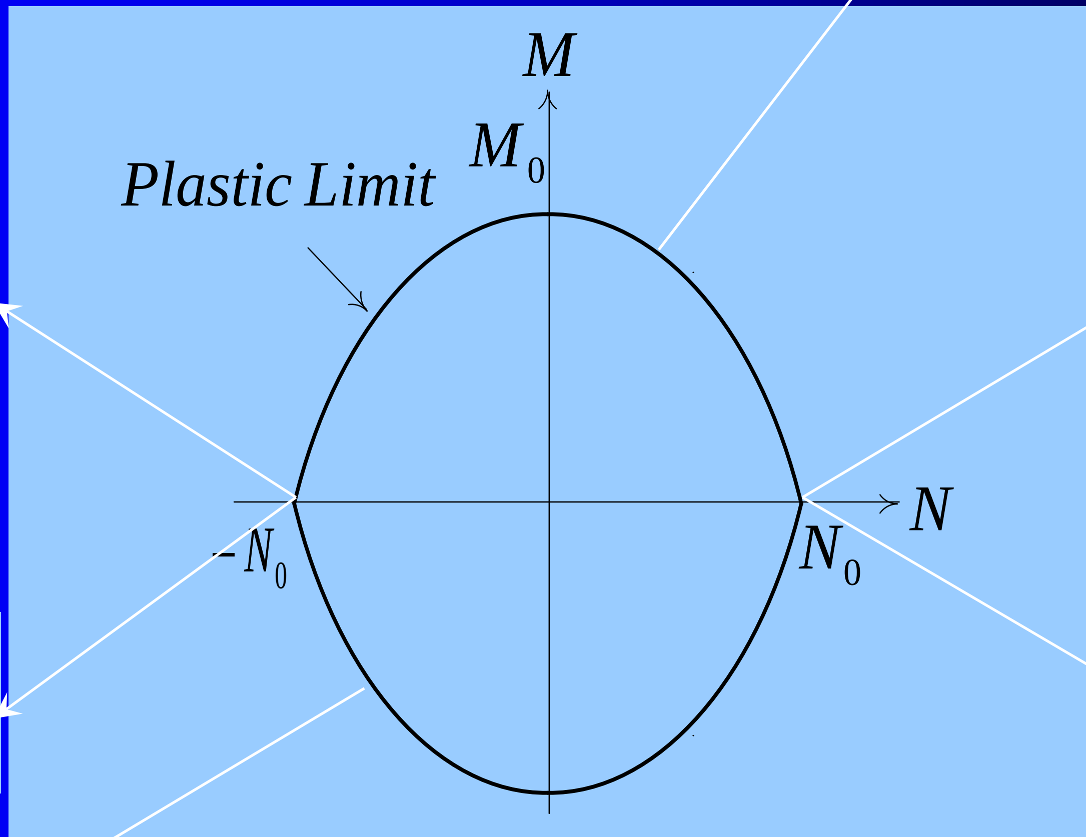
Plastic Limit of General Cross Section With Elastic Perfectly Plastic Material (continue…):

$$\frac{dM}{dN} = z_n$$

$$dz_n = \frac{d^2 M}{dN^2} < 0$$

$$M > 0$$
$$\left(z_n\right)_{\max} = h/2$$

$$M > 0$$
$$\left(z_n\right)_{\min} = -h/2$$



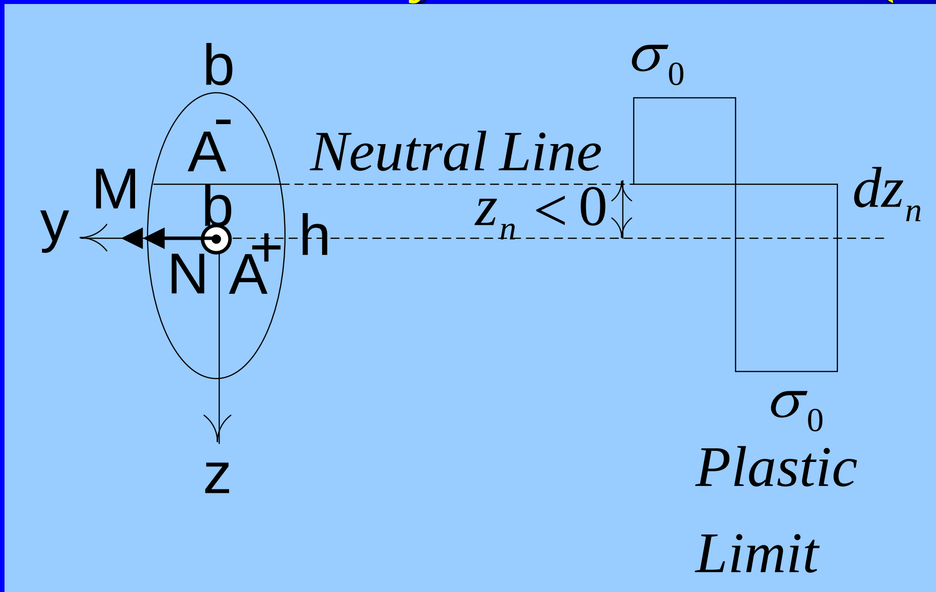
$$M < 0$$
$$\left(z_n\right)_{\min} = -h/2$$

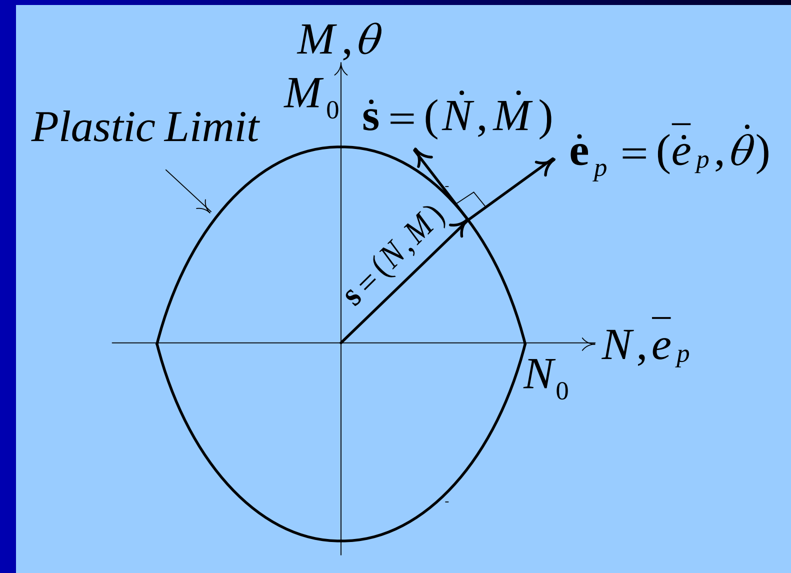
$$M < 0$$
$$\left(z_n\right)_{\max} = h/2$$

$$dz_n = \frac{d^2 M}{dN^2} > 0$$

Convexity Rule

Plastic Limit of General Cross Section With Elastic Perfectly Plastic Material (continue…):





$$\dot e_p = \dot{\bar{e}}_p + z\dot\theta$$

$$\dot N \dot{\bar{e}}_p + \dot M \dot\theta = 0$$

$$\dot e_{pn} = \dot{\bar{e}}_p + z_n\dot\theta = 0$$

$$\begin{bmatrix} \dot N & \dot M \end{bmatrix} \begin{Bmatrix} \dot{\bar{e}}_p \\ \dot\theta \end{Bmatrix} = 0$$

$$\dot{\bar{e}}_p + \frac{\dot M}{\dot N}\dot\theta = 0$$

$$\dot{\mathbf{s}}^T \dot{\mathbf{e}}_p = 0$$

Normality Rule

$$Plastic\ Function: f(\mathbf{s}) = 0$$

$$\dot{\mathbf{e}}_p = \dot\lambda \frac{\partial f}{\partial \mathbf{s}} = \dot\lambda \mathbf{f}_{\mathbf{s}}$$

$$\begin{Bmatrix} \dot{\bar{e}}_p \\ \dot\theta \end{Bmatrix} = \dot\lambda \begin{Bmatrix} f_{,N} \\ f_{,M} \end{Bmatrix}$$

Rigid-Perfectly Plastic:

$$Loading - Unloading\ Criteria:$$

$$f(\mathbf{s}) < 0 : Rigid\ Region$$

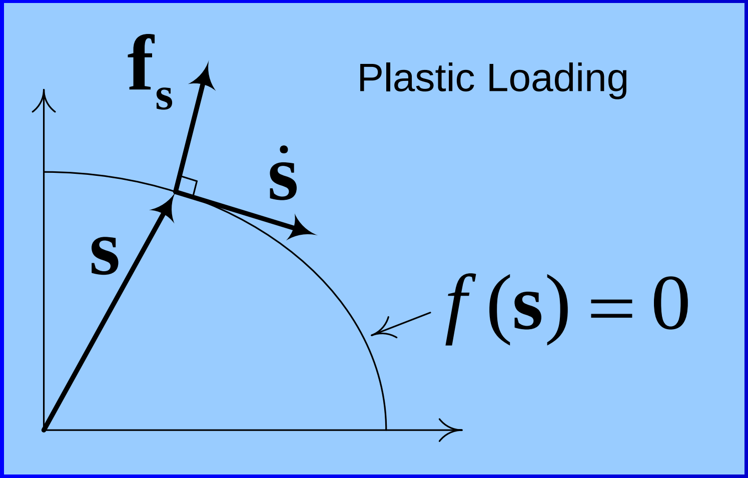
$$f(\mathbf{s}) = 0 \quad \& \quad \dot{f} = \mathbf{f}_{\mathbf{s}}^{T}\dot{\mathbf{s}} < 0 : Rigid\ Unloading$$

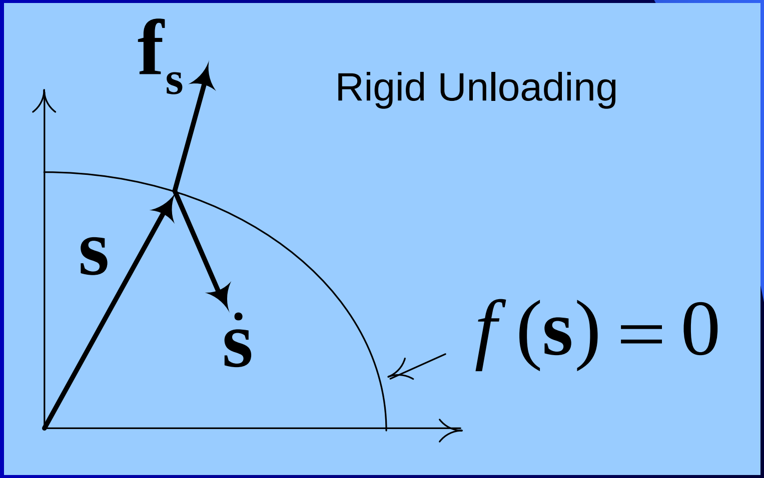
$$f(\mathbf{s}) = 0 \quad \& \quad \dot{f} = \mathbf{f}_{\mathbf{s}}^{T}\dot{\mathbf{s}} = 0 : Plastic\ Loading$$

$$Karush - Kuhn - Tucker \quad Condition:$$

$$\begin{cases} f(\mathbf{s}) \le 0 \\ \dot{\lambda} \ge 0 \\ \dot{\lambda}\mathbf{f}(\mathbf{s}) = 0 \end{cases}$$

$$: Consistency\ Condition$$





Incremental Analysis for Beam-Columns:

$$\begin{Bmatrix} \dot{\mathbf{s}}_I \\ \dot{\mathbf{s}}_J \end{Bmatrix} = \begin{bmatrix} \mathbf{K}_{II} & \mathbf{K}_{IJ} \\ \mathbf{K}_{JI} & \mathbf{K}_{JJ} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{d}}_I \\ \dot{\mathbf{d}}_J \end{Bmatrix}$$

$$\begin{cases} \dot{\mathbf{s}}_I = \dot{\mathbf{s}}_i \\ \dot{\mathbf{s}}_J = \dot{\mathbf{s}}_j \end{cases}$$

$$\begin{Bmatrix} \dot{\mathbf{s}}_i \\ \dot{\mathbf{s}}_j \end{Bmatrix} = \begin{bmatrix} \mathbf{K}_{II} & \mathbf{K}_{IJ} \\ \mathbf{K}_{JI} & \mathbf{K}_{JJ} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{d}}_I \\ \dot{\mathbf{d}}_J \end{Bmatrix}$$

-Rigid Connection:

$$\begin{cases} \dot{\mathbf{d}}_I = \dot{\mathbf{d}}_i \\ \dot{\mathbf{d}}_J = \dot{\mathbf{d}}_j \end{cases}$$

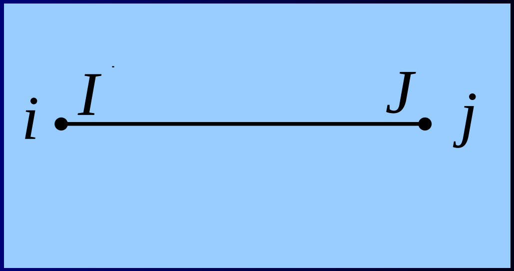
$$\begin{Bmatrix} \dot{\mathbf{s}}_i \\ \dot{\mathbf{s}}_j \end{Bmatrix} = \begin{bmatrix} \mathbf{K}_{II} & \mathbf{K}_{IJ} \\ \mathbf{K}_{JI} & \mathbf{K}_{JJ} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{d}}_i \\ \dot{\mathbf{d}}_j \end{Bmatrix}$$

-Plastic Connection for End I :

$$\dot{\mathbf{d}}_J = \dot{\mathbf{d}}_j$$

$$\begin{Bmatrix} \dot{\mathbf{s}}_i \\ \dot{\mathbf{s}}_j \end{Bmatrix} = \begin{bmatrix} \mathbf{K}_{II} & \mathbf{K}_{IJ} \\ \mathbf{K}_{JI} & \mathbf{K}_{JJ} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{d}}_I \\ \dot{\mathbf{d}}_j \end{Bmatrix}$$

-Plastic Connection for End I (continue…):



$$\begin{Bmatrix} \dot{\mathbf{s}}_i \\ \dot{\mathbf{s}}_j \end{Bmatrix} = \begin{bmatrix} \mathbf{K}_{II} & \mathbf{K}_{IJ} \\ \mathbf{K}_{JI} & \mathbf{K}_{JJ} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{d}}_I \\ \dot{\mathbf{d}}_j \end{Bmatrix}$$

$$\dot{\mathbf{s}}_i = \mathbf{K}_{II}\dot{\mathbf{d}}_I + \mathbf{K}_{IJ}\dot{\mathbf{d}}_j$$

$$\dot{\mathbf{e}}_{pi} = \dot{\lambda}\mathbf{f}_{\mathbf{si}} \qquad \dot{\mathbf{d}}_I - \dot{\mathbf{d}}_i = \dot{\lambda}\mathbf{f}_{\mathbf{si}} \qquad \dot{\mathbf{d}}_I = \dot{\mathbf{d}}_i + \dot{\lambda}\mathbf{f}_{\mathbf{si}}$$

$$\dot{\mathbf{s}}_i = \mathbf{K}_{II}\left(\dot{\mathbf{d}}_i + \dot{\lambda}\mathbf{f}_{\mathbf{si}}\right) + \mathbf{K}_{IJ}\dot{\mathbf{d}}_j$$

$$\mathbf{f}_{\mathbf{si}}^T\dot{\mathbf{s}}_i = 0 \qquad \mathbf{f}_{\mathbf{si}}^T\mathbf{K}_{II}\left(\dot{\mathbf{d}}_i + \dot{\lambda}\mathbf{f}_{\mathbf{si}}\right) + \mathbf{f}_{\mathbf{si}}^T\mathbf{K}_{IJ}\dot{\mathbf{d}}_j = 0$$

$$\mathbf{f}_{\mathbf{si}}^T\mathbf{K}_{II}\dot{\mathbf{d}}_i + \dot{\lambda}\mathbf{f}_{\mathbf{si}}^T\mathbf{K}_{II}\mathbf{f}_{\mathbf{si}} + \mathbf{f}_{\mathbf{si}}^T\mathbf{K}_{IJ}\dot{\mathbf{d}}_j = 0$$

$$\dot{\lambda} = -\frac{\mathbf{f}_{\mathbf{si}}^T\mathbf{K}_{II}\dot{\mathbf{d}}_i + \mathbf{f}_{\mathbf{si}}^T\mathbf{K}_{IJ}\dot{\mathbf{d}}_j}{\mathbf{f}_{\mathbf{si}}^T\mathbf{K}_{II}\mathbf{f}_{\mathbf{si}}}$$

$$\dot{\mathbf{d}}_I = \dot{\mathbf{d}}_i - \mathbf{f}_{\mathbf{si}}\frac{\mathbf{f}_{\mathbf{si}}^T\mathbf{K}_{II}\dot{\mathbf{d}}_i + \mathbf{f}_{\mathbf{si}}^T\mathbf{K}_{IJ}\dot{\mathbf{d}}_j}{\mathbf{f}_{\mathbf{si}}^T\mathbf{K}_{II}\mathbf{f}_{\mathbf{si}}}$$

-Plastic Connection for End I (continue…):

$$\begin{Bmatrix} \dot{\mathbf{s}}_i \\ \dot{\mathbf{s}}_j \end{Bmatrix} = \begin{bmatrix} \mathbf{K}_{II} & \mathbf{K}_{IJ} \\ \mathbf{K}_{JI} & \mathbf{K}_{JJ} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{d}}_I \\ \dot{\mathbf{d}}_j \end{Bmatrix}$$

$$\dot{\mathbf{d}}_I = \dot{\mathbf{d}}_i - \mathbf{f}_{si} \frac{\mathbf{f}_{si}^T \mathbf{K}_{II} \dot{\mathbf{d}}_i + \mathbf{f}_{si}^T \mathbf{K}_{IJ} \dot{\mathbf{d}}_j}{\mathbf{f}_{si}^T \mathbf{K}_{II} \mathbf{f}_{si}}$$

$$\dot{\mathbf{s}}_i = \mathbf{K}_{II} \dot{\mathbf{d}}_I + \mathbf{K}_{IJ} \dot{\mathbf{d}}_j$$

$$\dot{\mathbf{s}}_i = \mathbf{K}_{II} \left(\dot{\mathbf{d}}_i - \mathbf{f}_{si} \frac{\mathbf{f}_{si}^T \mathbf{K}_{II} \dot{\mathbf{d}}_i + \mathbf{f}_{si}^T \mathbf{K}_{IJ} \dot{\mathbf{d}}_j}{\mathbf{f}_{si}^T \mathbf{K}_{II} \mathbf{f}_{si}} \right) + \mathbf{K}_{IJ} \dot{\mathbf{d}}_j$$

$$\dot{\mathbf{s}}_i = \left(\mathbf{K}_{II} - \frac{\mathbf{K}_{II} \mathbf{f}_{si} \mathbf{f}_{si}^T \mathbf{K}_{II}}{\mathbf{f}_{si}^T \mathbf{K}_{II} \mathbf{f}_{si}} \right) \dot{\mathbf{d}}_i + \left(\mathbf{K}_{IJ} - \frac{\mathbf{K}_{II} \mathbf{f}_{si} \mathbf{f}_{si}^T \mathbf{K}_{IJ}}{\mathbf{f}_{si}^T \mathbf{K}_{II} \mathbf{f}_{si}} \right) \dot{\mathbf{d}}_j$$

-Plastic Connection for End I (continue…):

$$\begin{Bmatrix} \dot{\mathbf{s}}_i \\ \dot{\mathbf{s}}_j \end{Bmatrix} = \begin{bmatrix} \mathbf{K}_{II} & \mathbf{K}_{IJ} \\ \mathbf{K}_{JI} & \mathbf{K}_{JJ} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{d}}_I \\ \dot{\mathbf{d}}_j \end{Bmatrix}$$

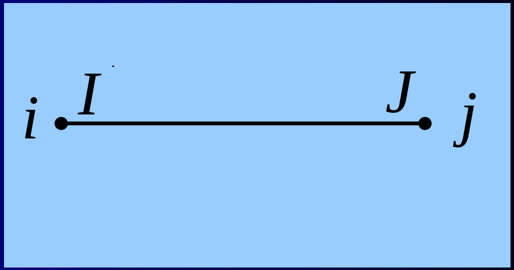
$$\dot{\mathbf{d}}_I = \dot{\mathbf{d}}_i - \mathbf{f}_{si} \frac{\mathbf{f}_{si}^T \mathbf{K}_{II} \dot{\mathbf{d}}_i + \mathbf{f}_{si}^T \mathbf{K}_{IJ} \dot{\mathbf{d}}_j}{\mathbf{f}_{si}^T \mathbf{K}_{II} \mathbf{f}_{si}}$$

$$\dot{\mathbf{s}}_j = \mathbf{K}_{JI} \dot{\mathbf{d}}_I + \mathbf{K}_{JJ} \dot{\mathbf{d}}_j$$

$$\dot{\mathbf{s}}_j = \mathbf{K}_{JI} \left(\dot{\mathbf{d}}_i - \mathbf{f}_{si} \frac{\mathbf{f}_{si}^T \mathbf{K}_{II} \dot{\mathbf{d}}_i + \mathbf{f}_{si}^T \mathbf{K}_{IJ} \dot{\mathbf{d}}_j}{\mathbf{f}_{si}^T \mathbf{K}_{II} \mathbf{f}_{si}} \right) + \mathbf{K}_{JJ} \dot{\mathbf{d}}_j$$

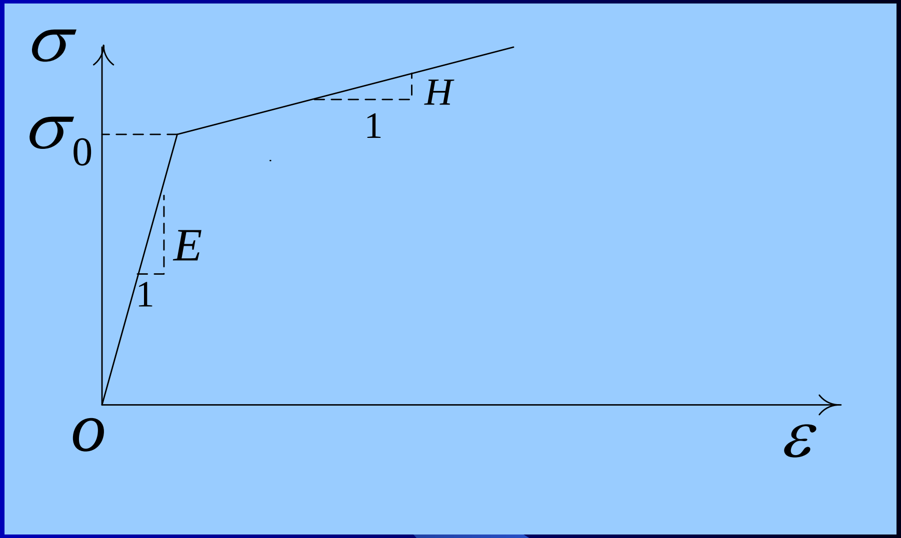
$$\dot{\mathbf{s}}_j = \left(\mathbf{K}_{JI} - \frac{\mathbf{K}_{JI} \mathbf{f}_{si} \mathbf{f}_{si}^T \mathbf{K}_{II}}{\mathbf{f}_{si}^T \mathbf{K}_{II} \mathbf{f}_{si}} \right) \dot{\mathbf{d}}_i + \left(\mathbf{K}_{JJ} - \frac{\mathbf{K}_{JI} \mathbf{f}_{si} \mathbf{f}_{si}^T \mathbf{K}_{IJ}}{\mathbf{f}_{si}^T \mathbf{K}_{II} \mathbf{f}_{si}} \right) \dot{\mathbf{d}}_j$$

-Plastic Connection for End I (continue…):

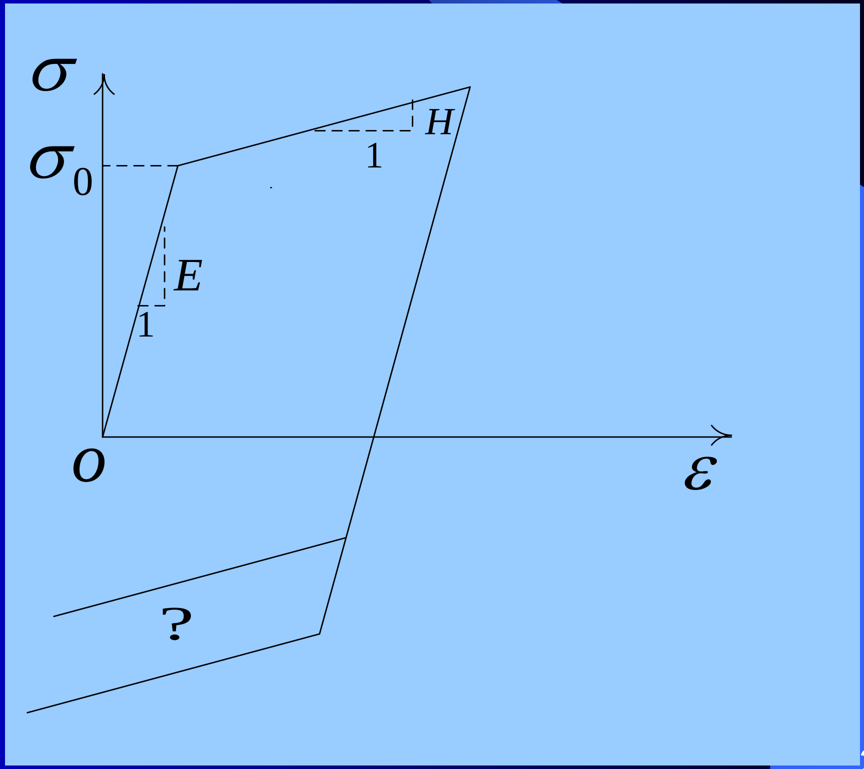


$$\begin{Bmatrix} \dot{\mathbf{s}}_i \\ \dot{\mathbf{s}}_j \end{Bmatrix} = \begin{bmatrix} \mathbf{K}_{II} - \dfrac{\mathbf{K}_{II}\mathbf{f}_{si}\mathbf{f}_{si}^T\mathbf{K}_{II}}{\mathbf{f}_{si}^T\mathbf{K}_{II}\mathbf{f}_{si}} & \mathbf{K}_{IJ} - \dfrac{\mathbf{K}_{II}\mathbf{f}_{si}\mathbf{f}_{si}^T\mathbf{K}_{IJ}}{\mathbf{f}_{si}^T\mathbf{K}_{II}\mathbf{f}_{si}} \\ \mathbf{K}_{JI} - \dfrac{\mathbf{K}_{JI}\mathbf{f}_{si}\mathbf{f}_{si}^T\mathbf{K}_{II}}{\mathbf{f}_{si}^T\mathbf{K}_{II}\mathbf{f}_{si}} & \mathbf{K}_{JJ} - \dfrac{\mathbf{K}_{JI}\mathbf{f}_{si}\mathbf{f}_{si}^T\mathbf{K}_{IJ}}{\mathbf{f}_{si}^T\mathbf{K}_{II}\mathbf{f}_{si}} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{d}}_i \\ \dot{\mathbf{d}}_j \end{Bmatrix}$$

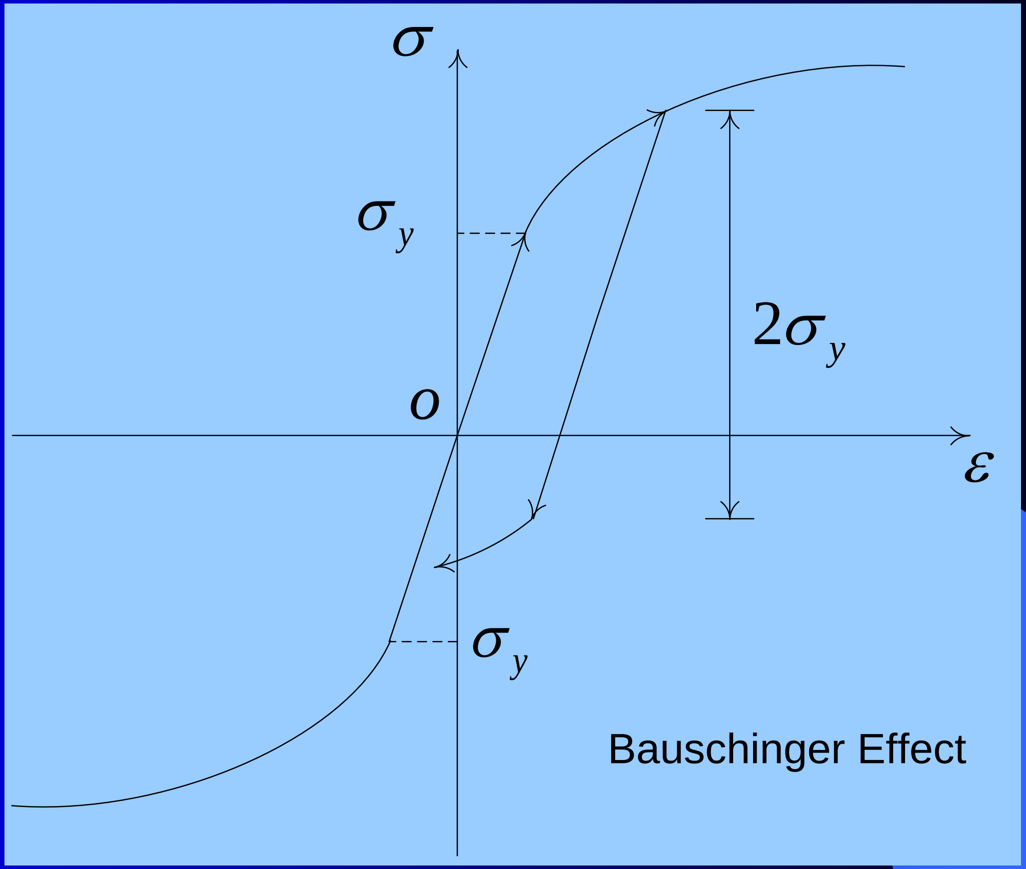
Elastic-Plastic Behavior
in Monotonic Loading:



Elastic-Plastic Behavior
in Reversed Loading:



Bauschinger Effect in Reversed Loadings



Hardening Parameters

$$Equivalent\ Plastic\ Strain:$$

$$\varepsilon^{p} = \int \left| d\varepsilon_{p} \right| = \int \left(d\varepsilon_{p} d\varepsilon_{p} \right)^{\frac{1}{2}}$$

$$Plastic\ Work$$

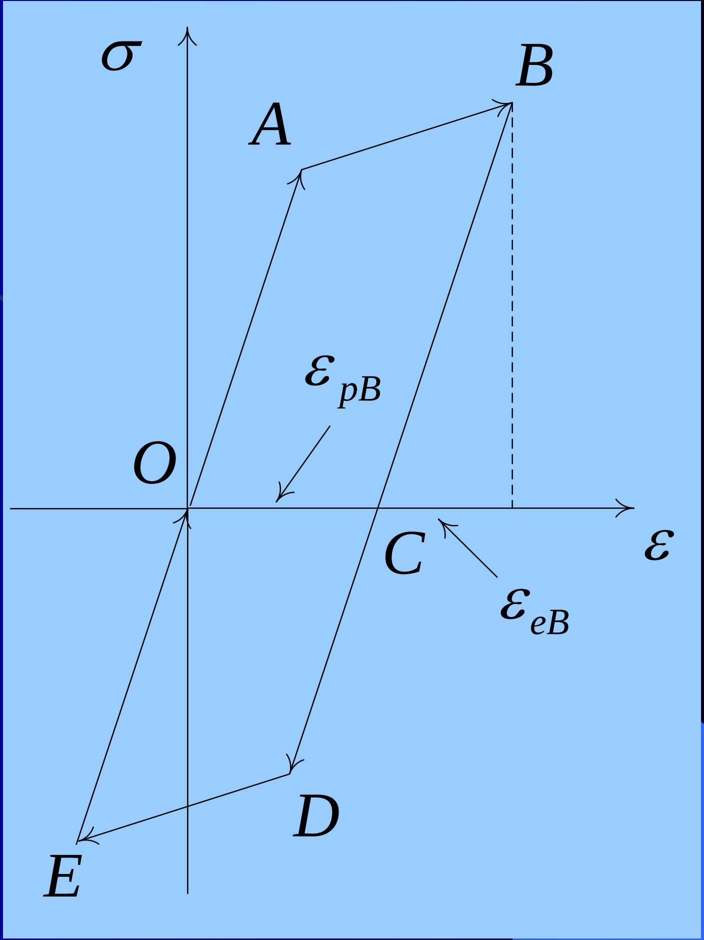
$$W_{p} = \int \sigma d\varepsilon_{p}$$

$$Plastic\ Strain$$

$$\varepsilon_{p} = \int d\varepsilon_{p}$$

$$Path: OAB \begin{cases} \varepsilon^{p} = \varepsilon_{pB} \\ W_{p} = Area\ of\ OABCO \\ \varepsilon_{p} = \varepsilon_{pB} \end{cases}$$

$$Path: OABCDEO \begin{cases} \varepsilon^{p} = 2\varepsilon_{pB} \\ W_{p} = Area\ of\ OABCDEO \\ \varepsilon_{p} = 0 \end{cases}$$



Isotropic Hardening:

$$Yield\ Condition:$$

$$\left|\sigma\right| = \sigma_Y\left(\kappa\right)$$

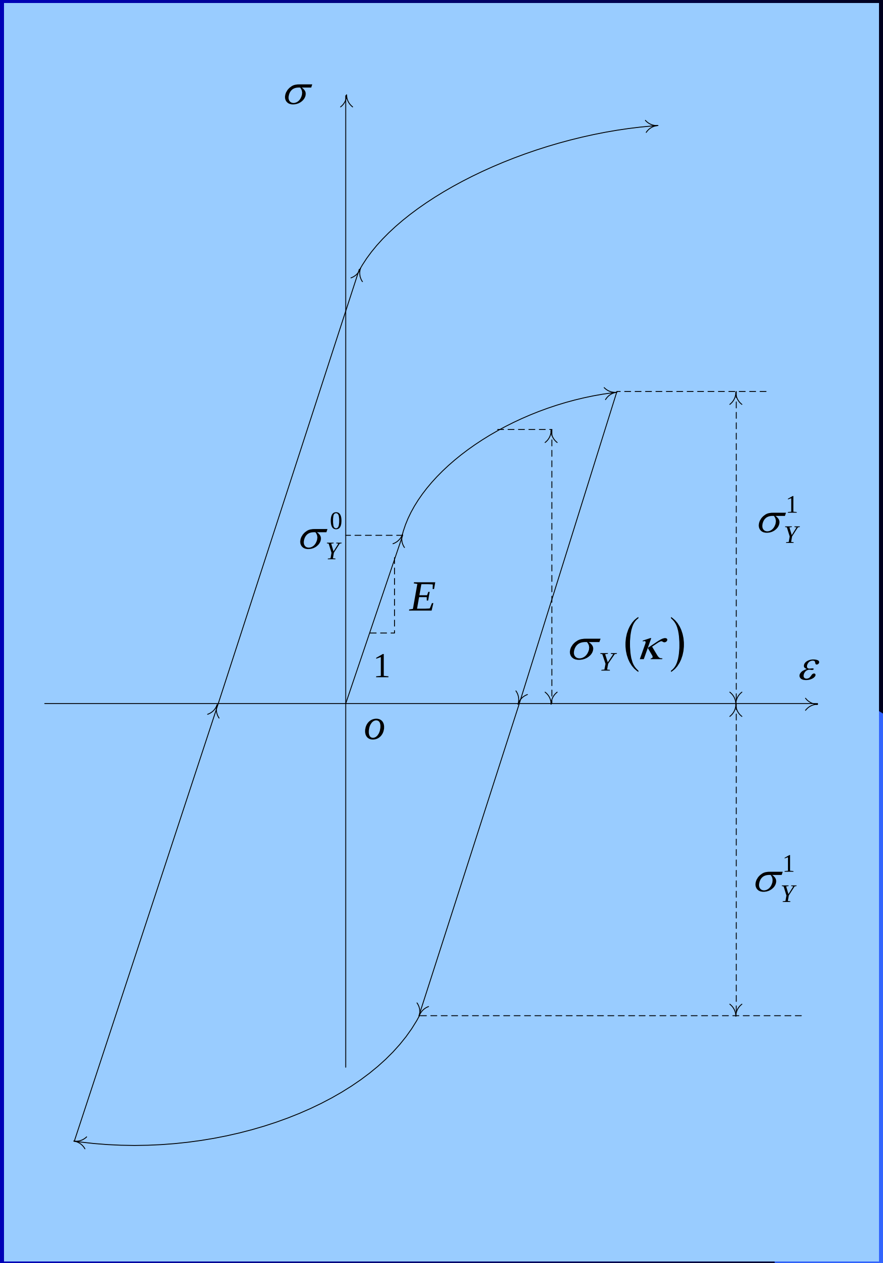
$$\sigma_Y\left(\kappa\right) = \sigma_Y^0 + h(\kappa)$$

κ : *Accumulative Hardening Parameter*

$$\kappa = \begin{cases} \varepsilon^p \\ or \\ W_p \end{cases}$$

$$\kappa = \varepsilon^p \ : Strain\ Hardening$$

$$\kappa = W_p \ : Work\ Hardening$$



Linear Isotropic Strain Hardening:

Yield Condition :

$$|\sigma| = \sigma_Y\left(\varepsilon^p\right)$$

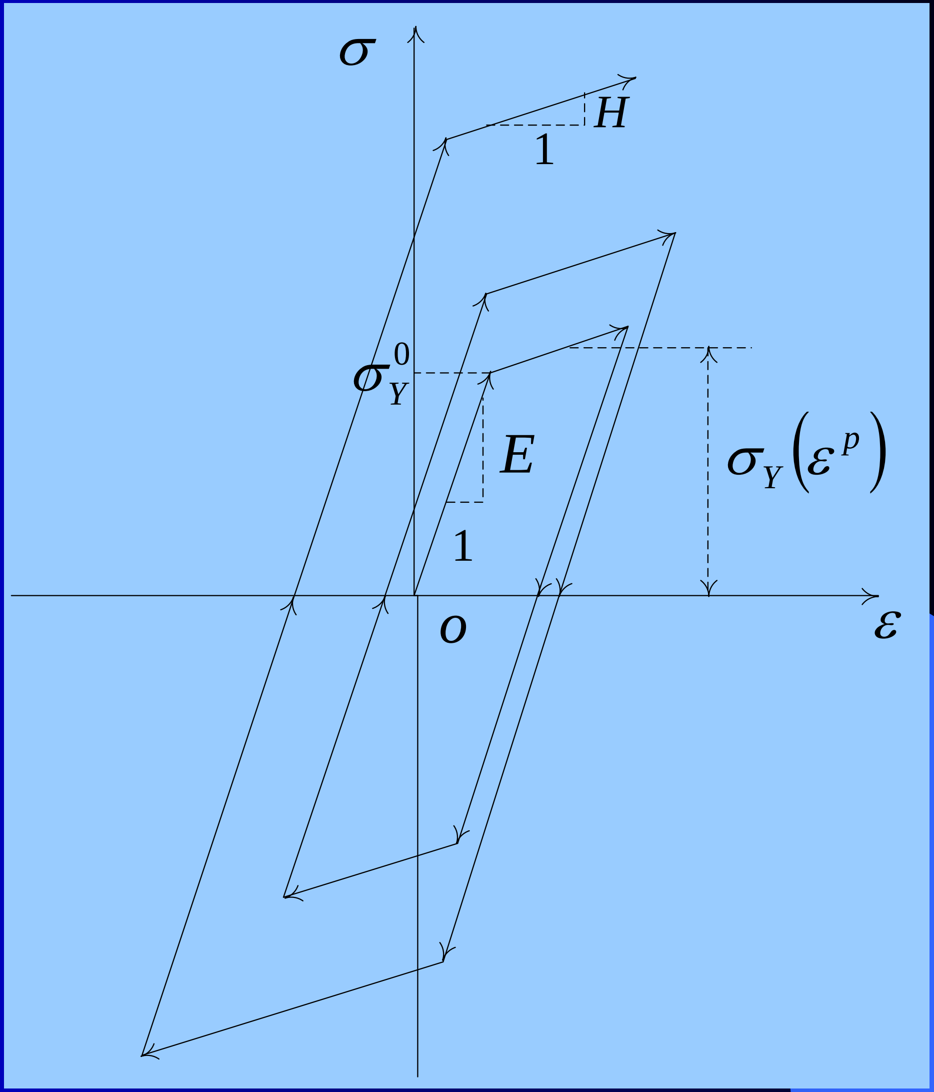
$$\sigma_Y\left(\varepsilon^p\right) = \sigma_Y^0 + E_p\varepsilon^p$$

$$\dot{\sigma}_Y\left(\varepsilon^p\right) = E_p\dot{\varepsilon}^p$$

$$\frac{\dot{\sigma}_Y\left(\varepsilon^p\right)}{\dot{\varepsilon}^p} = E_p$$

$$E_t = \frac{EE_P}{E+E_p} = H = Constant$$

$$E_p = \frac{EE_t}{E-E_t}$$

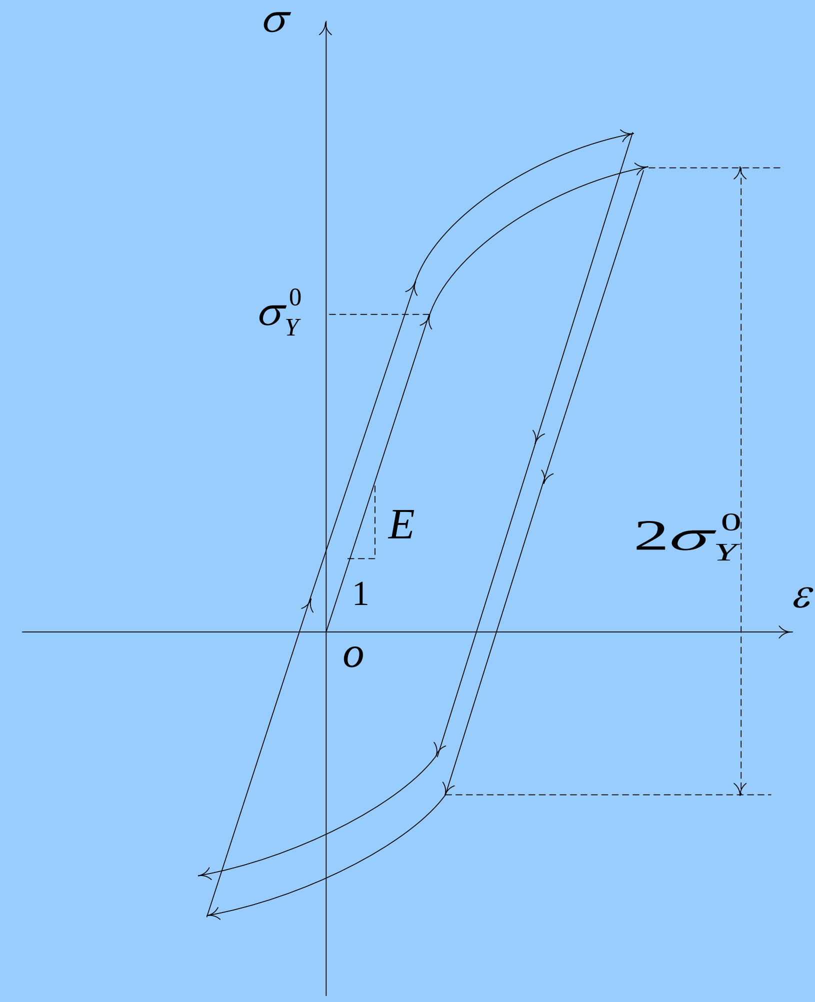


Kinematic Hardening:

$$\textit{Yield Condition}:$$
$$\left|\sigma - \alpha\left(\varepsilon_p\right)\right| = \sigma_{Y0}$$

$$\textit{Plastic Strain}$$
$$\varepsilon_p = \int d\varepsilon_p$$

$$\alpha : \textit{Back Stress}$$



Linear Kinematic Hardening:

$$\text{Yield Condition}:$$
$$\left|\sigma - \alpha\left(\varepsilon_p\right)\right| = \sigma_{Y0}$$

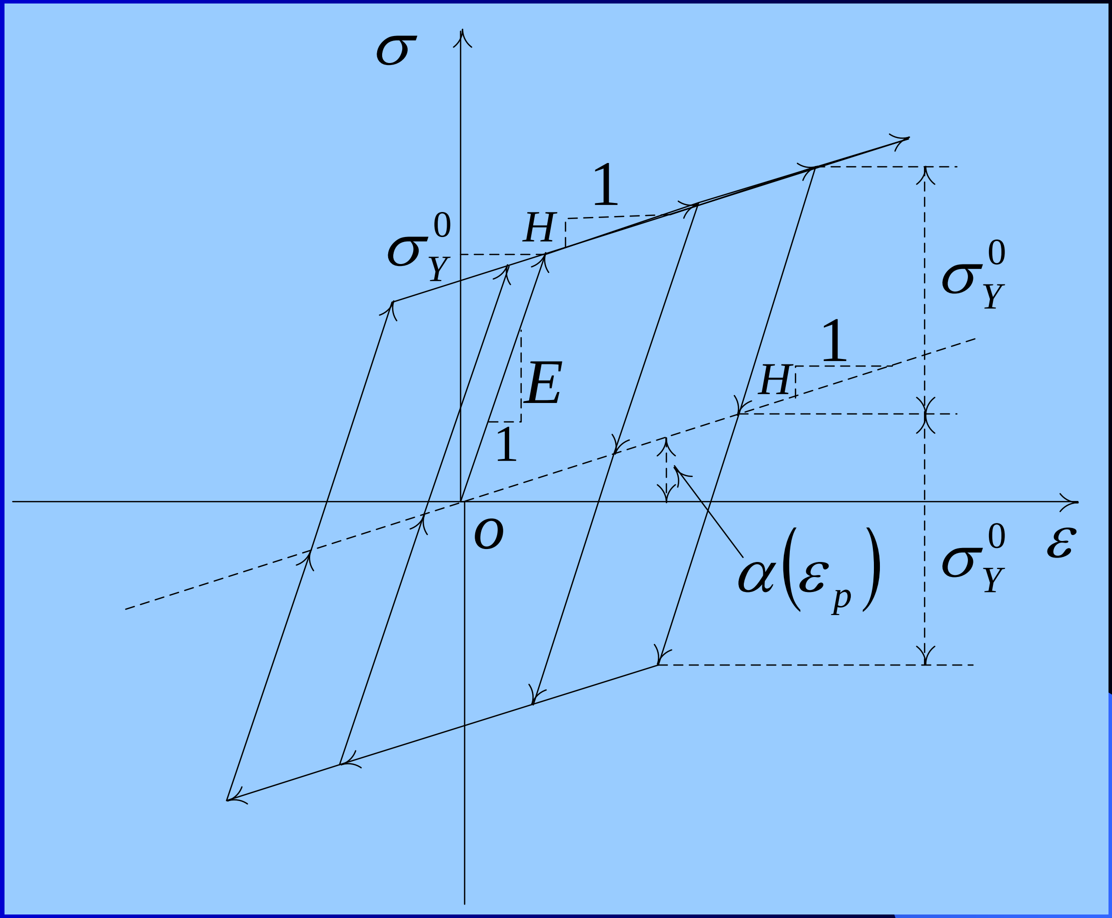
$$\alpha\left(\varepsilon_p\right) = E_p \varepsilon_p$$

$$\dot{\alpha}\left(\varepsilon_p\right) = E_p \dot{\varepsilon}_p$$

$$\frac{\dot{\alpha}\left(\varepsilon_p\right)}{\dot{\varepsilon}_p} = E_p$$

$$E_t = \frac{EE_P}{E + E_p} = H = Constant$$

$$E_p = \frac{EE_t}{E - E_t}$$



Mixed Hardening:

$$Yield\ Condition:$$

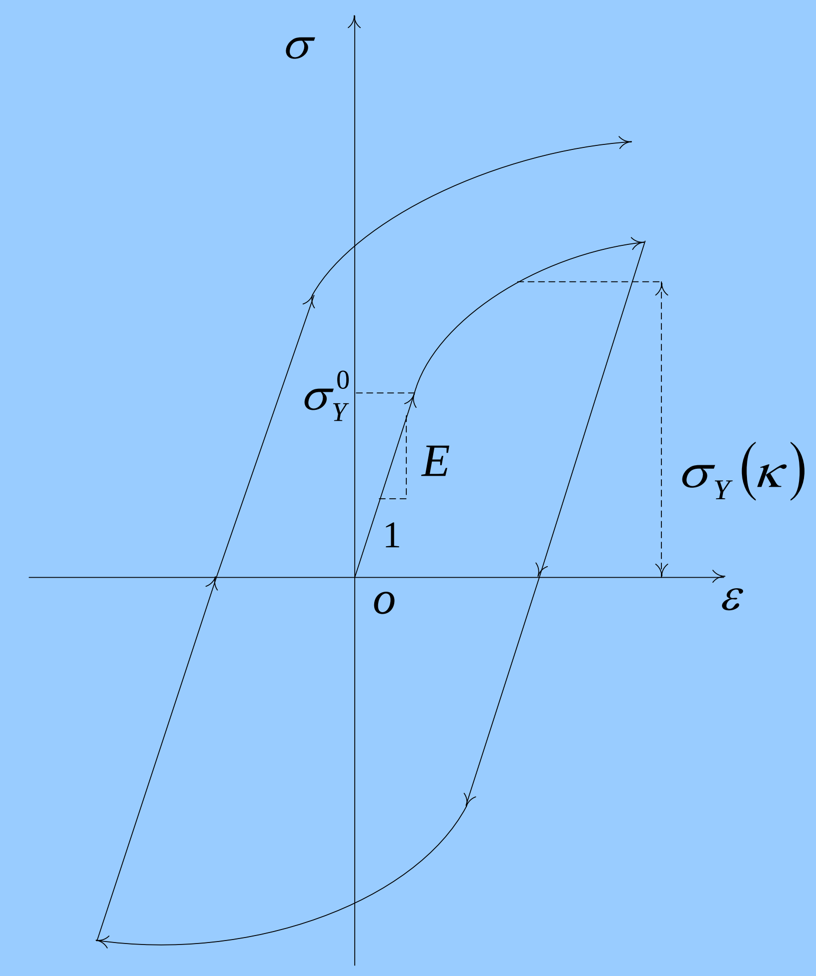
$$\left|\sigma - \alpha\left(\varepsilon_p\right)\right| = \sigma_Y\left(\kappa\right)$$

$$\sigma_Y\left(\kappa\right) = \sigma_Y^0 + h(\kappa)$$

$$\kappa : Accumulative$$

$$Hardening\ Parameter$$

$$\kappa = \begin{cases} \varepsilon^p \\ or \\ W_p \end{cases}$$



Mixed Linear Hardening

Yield Condition:

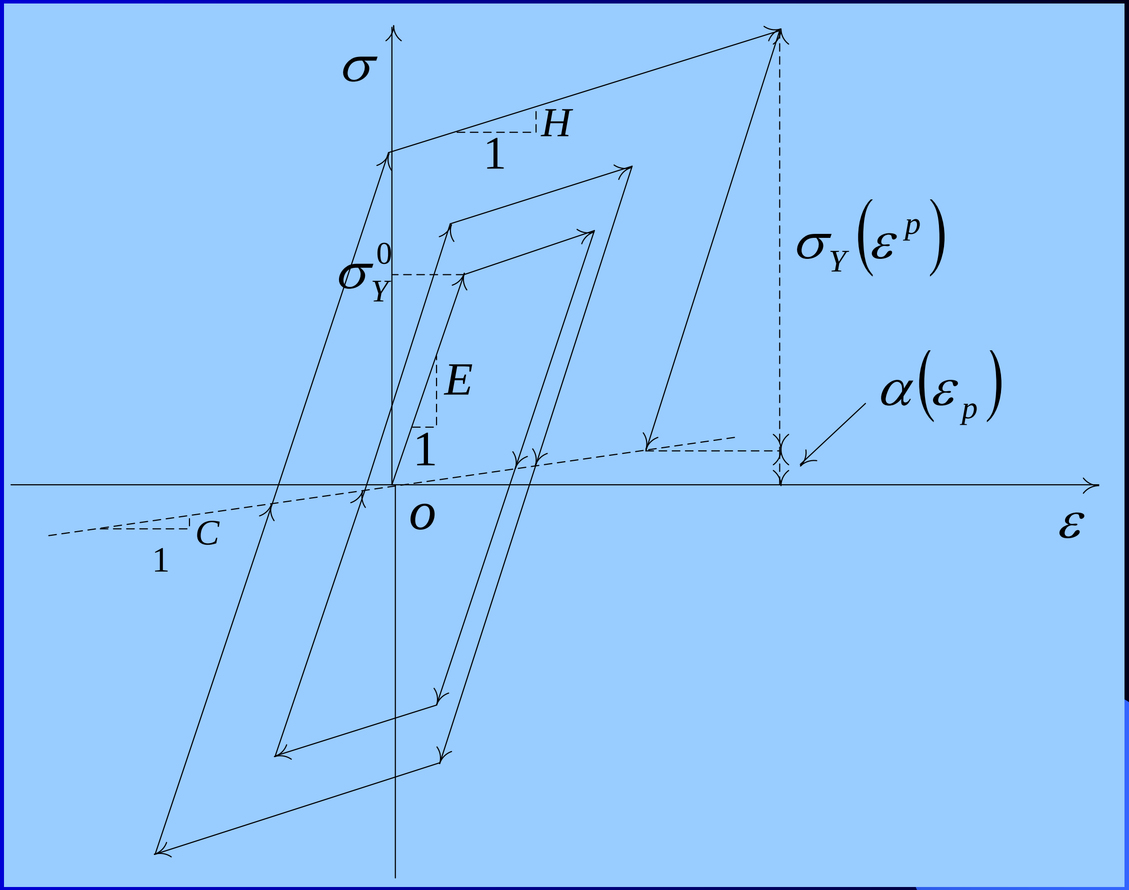
$$\left|\sigma - \alpha\left(\varepsilon_p\right)\right| = \sigma_Y\left(\varepsilon^p\right)$$

$$\alpha\left(\varepsilon_p\right) = C_p \varepsilon_p$$

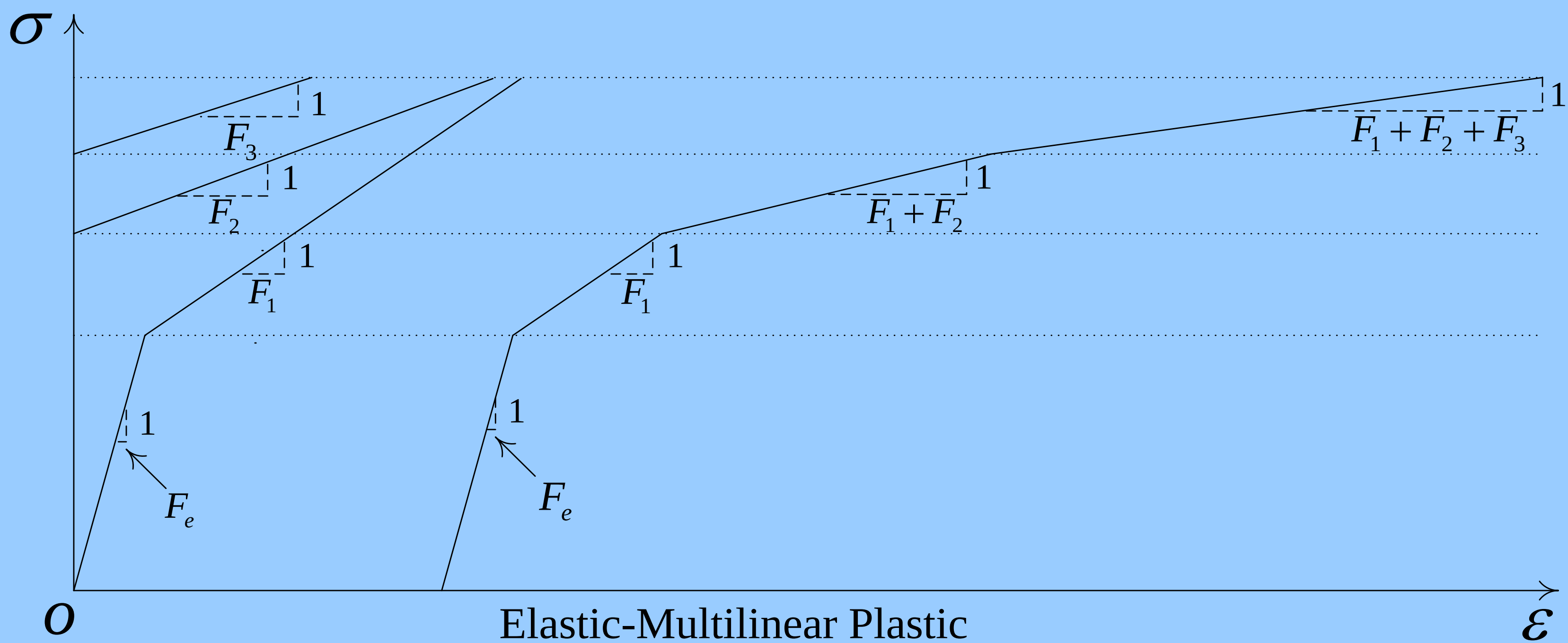
$$\sigma_Y\left(\varepsilon^p\right) = \sigma_Y^0 + \left(E_P - C_p\right)\varepsilon^p$$

$$C_p = \frac{EC}{E - C}$$

$$E_p = \frac{EH}{E - H}$$

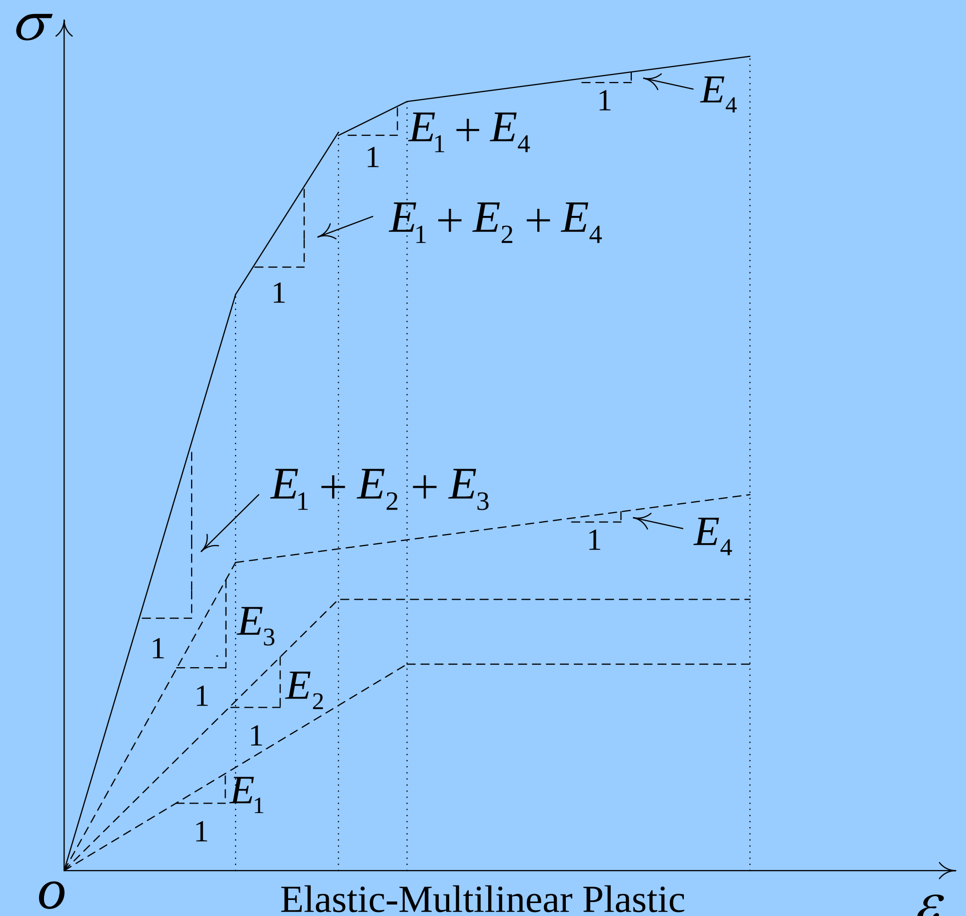


Multi-linear Stress Strain Relations



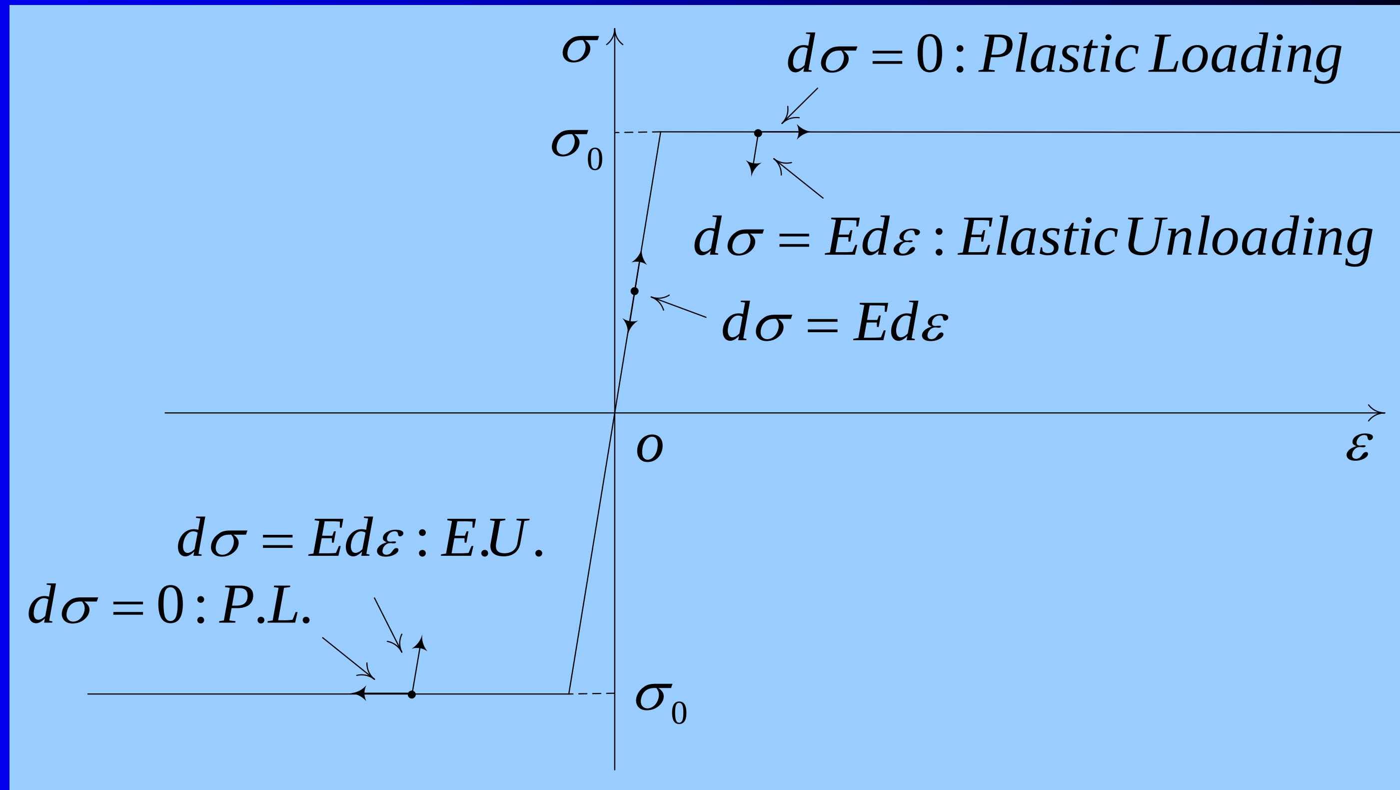
Elastic-Multilinear Plastic
(serial combination)

Multi-linear Stress Strain Relations (continue …)



Elastic-Multilinear Plastic
(parallel combination)

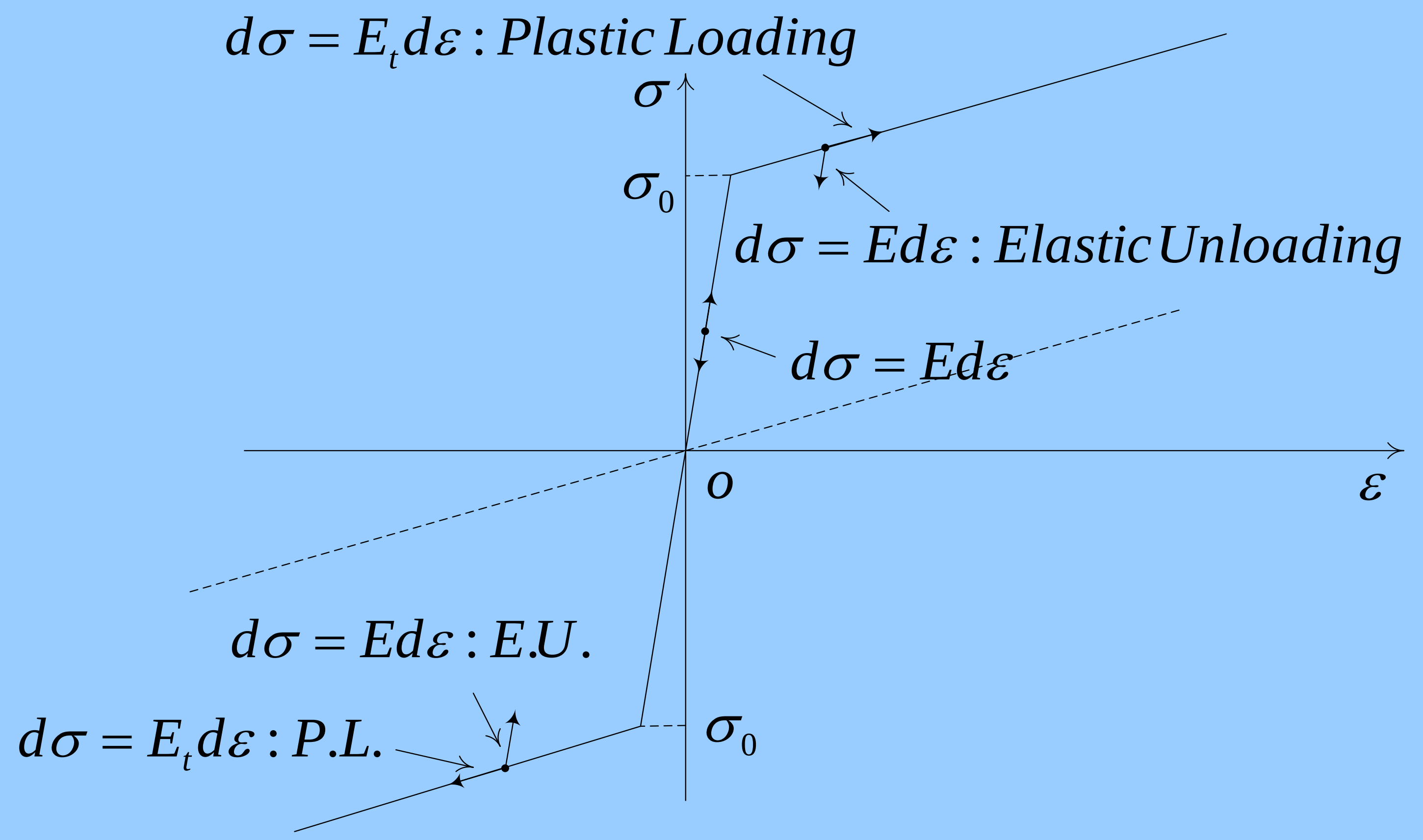
Loading – Unloading Criteria for Elastic Perfectly Plastic Behavior



$$Elastic\ Region:$$
$$d\sigma = E d\varepsilon$$

$$Yield\ Region:$$
$$\begin{cases} if\ \sigma d\sigma < 0 : Elastic\ Unloading \\ if\ \sigma d\sigma = 0 : Plastic\ Loading \end{cases}$$

Loading – Unloading Criteria for Elastic Hardening Plasticity



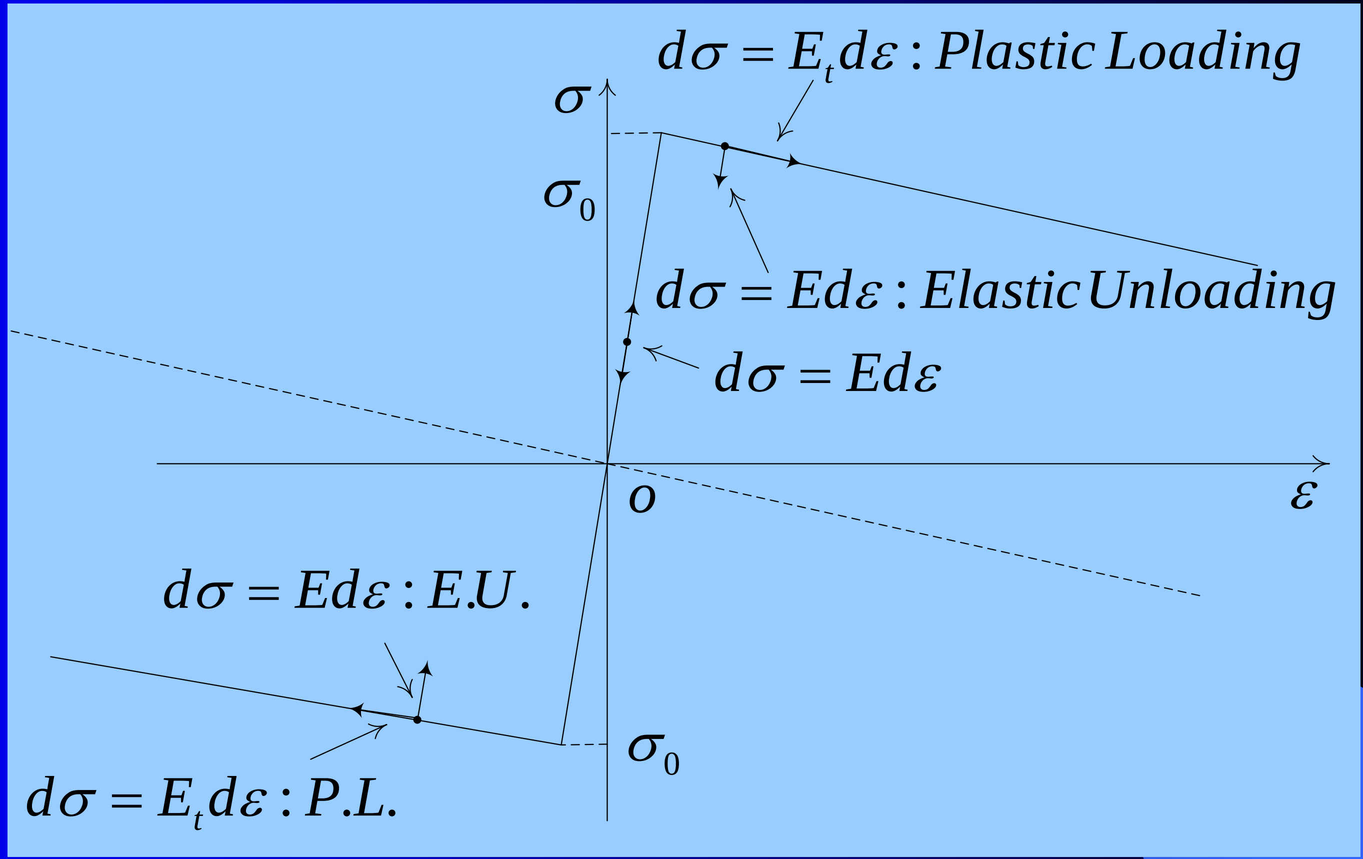
$Elastic\ Region:$

$d\sigma = E d\varepsilon$

$Yield\ Region:$

$$\begin{cases} if\ (\sigma - \alpha) d\sigma < 0 : Elastic\ Unloading \\ if\ (\sigma - \alpha) d\sigma > 0 : Plastic\ Loading \end{cases}$$

Loading – Unloading Criteria for Elastic Softening Plasticity



Elastic Region:

$$d\sigma = Ed\varepsilon$$

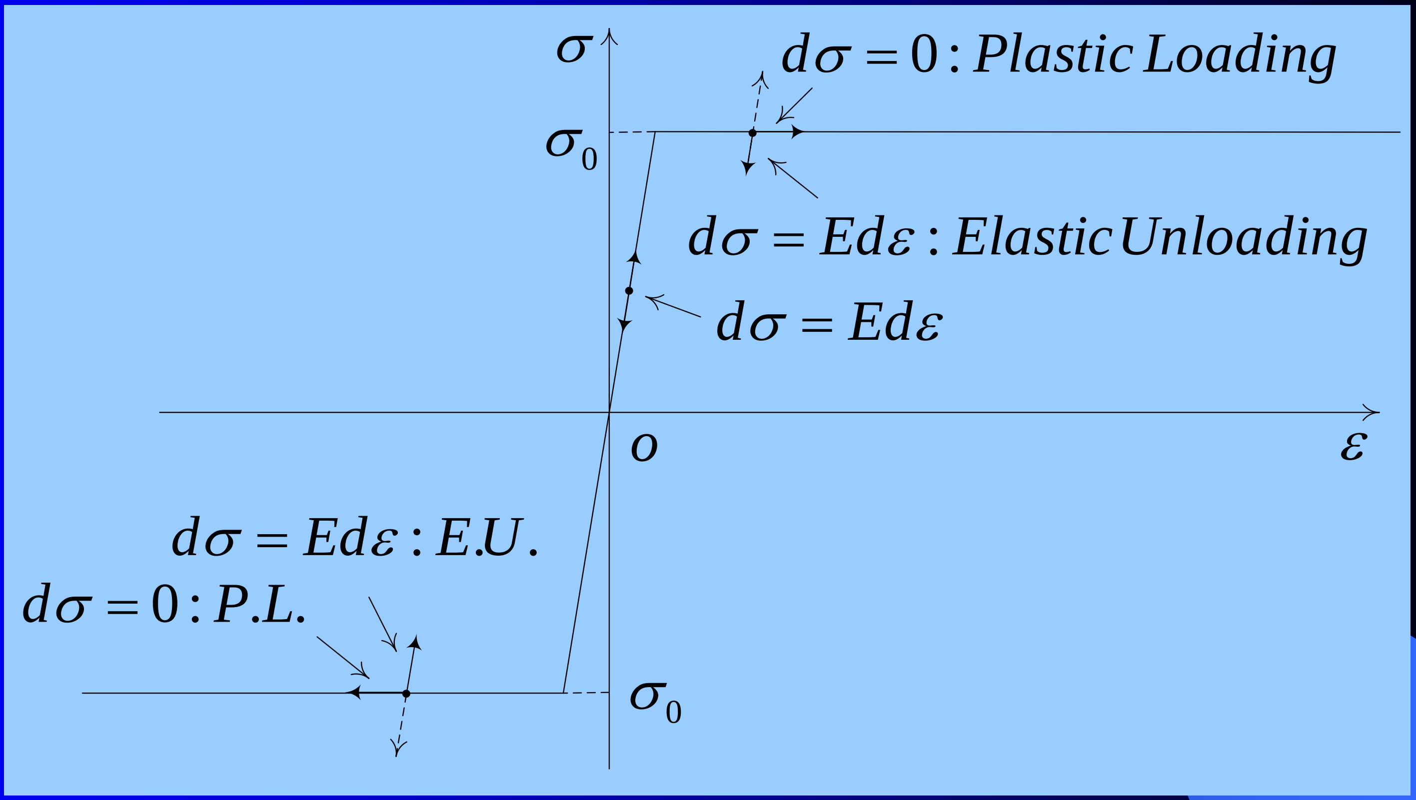
Yield Region:

$$\begin{cases} Elastic\,Unloading \\ Plastic\,Loading \end{cases} : (\sigma - \alpha)\,d\sigma < 0$$

Yield Region:

$$\begin{cases} if\ (\varepsilon - \varepsilon_p)\,d\varepsilon < 0 : Elastic\,Unloading \\ if\ (\varepsilon - \varepsilon_p)\,d\varepsilon > 0 : Plastic\,Loading \end{cases}$$

Elastic Predictor – Plastic Corrector for Elastic Perfectly Plastic Behavior

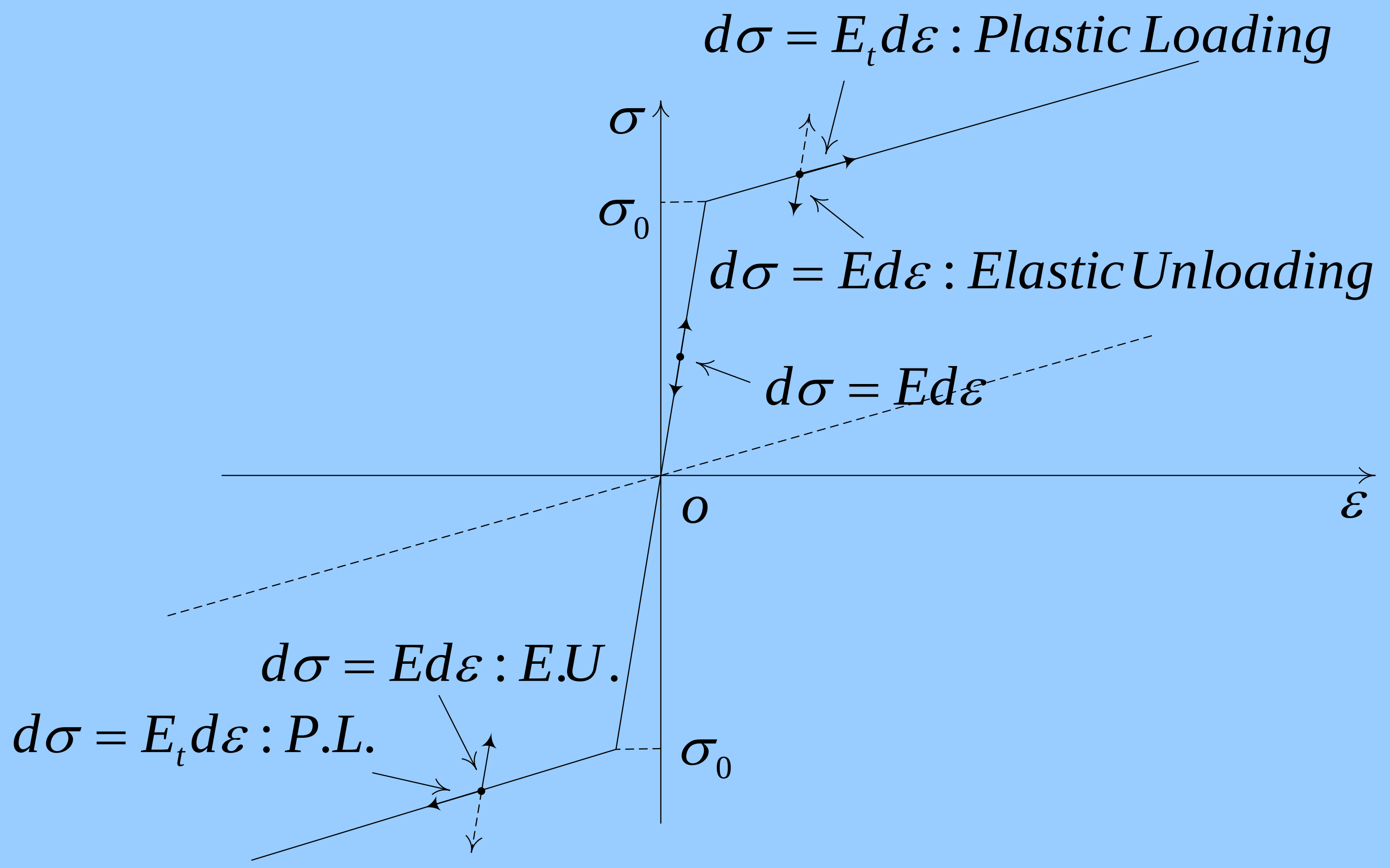


$$Elastic\ Region:$$
$$d\sigma = Ed\varepsilon$$

$$Yield\ Region:$$

$$d\sigma = Ed\varepsilon\left(Elastic\ Predictor\right) \rightarrow \begin{cases} if\ \sigma d\sigma < 0 : Elastic\ Unloading \\ else : Plastic\ Loading\left(Plastic\ Corrector\right) \end{cases}$$

Elastic Predictor – Plastic Corrector for Elastic Hardening Plasticity



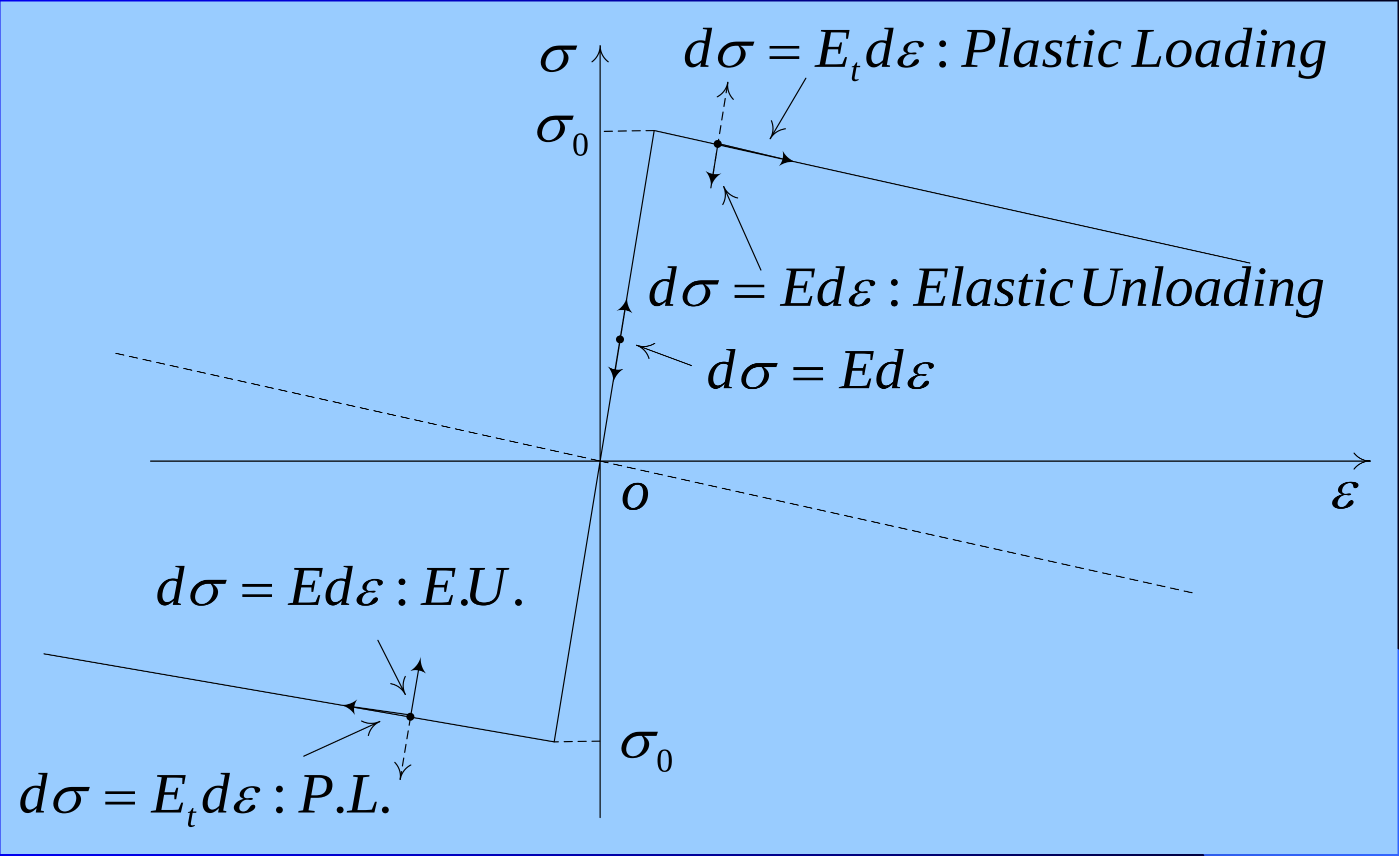
$Elastic\ Region:$

$$d\sigma = E d\varepsilon$$

$Yield\ Region:$

$$d\sigma = E d\varepsilon \left(Elastic\ Predictor\right) \rightarrow \begin{cases} if\ (\sigma - \alpha) d\sigma < 0 : Elastic\ Unloading \\ else : Plastic\ Loading\left(Plastic\ Corrector\right) \end{cases}$$

Elastic Predictor – Plastic Corrector for Elastic Softening Plasticity



$$Elastic\ Region: \quad d\sigma = Ed\varepsilon$$

$$Yield\ Region:$$

$$d\sigma = Ed\varepsilon\left(Elastic\ Predictor\right) \rightarrow \begin{cases} if\ (\sigma-\alpha)d\sigma < 0 : Elastic\ Unloading \\ else : Plastic\ Loading\left(Plastic\ Corrector\right) \end{cases}$$

Nonlinear Analysis

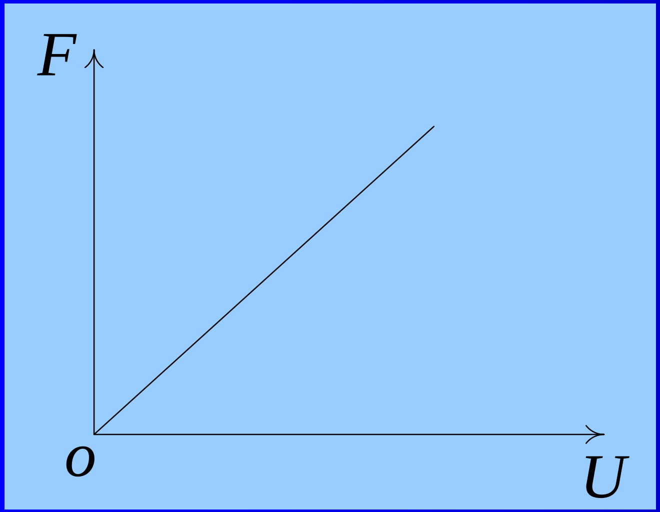
$$Equilibrium\ Equation: \mathbf{KU} = \mathbf{F}$$

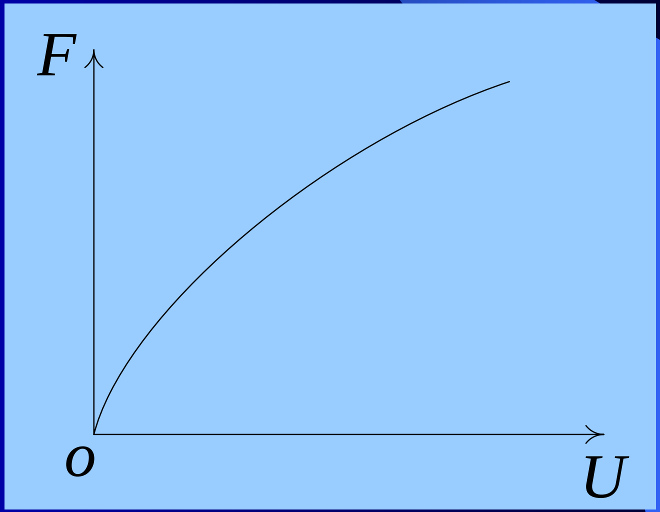
$$Linear\ Analysis: \mathbf{K}\ is\ constant$$

$$Solution: \mathbf{U} = \mathbf{K}^{-1}\mathbf{F}$$

$$Nonlinear\ Analysis: \mathbf{K} = \mathbf{K}(\mathbf{U})$$

$$Incremental\ Analysis: \mathbf{K}_t d\mathbf{U} = d\mathbf{F}$$



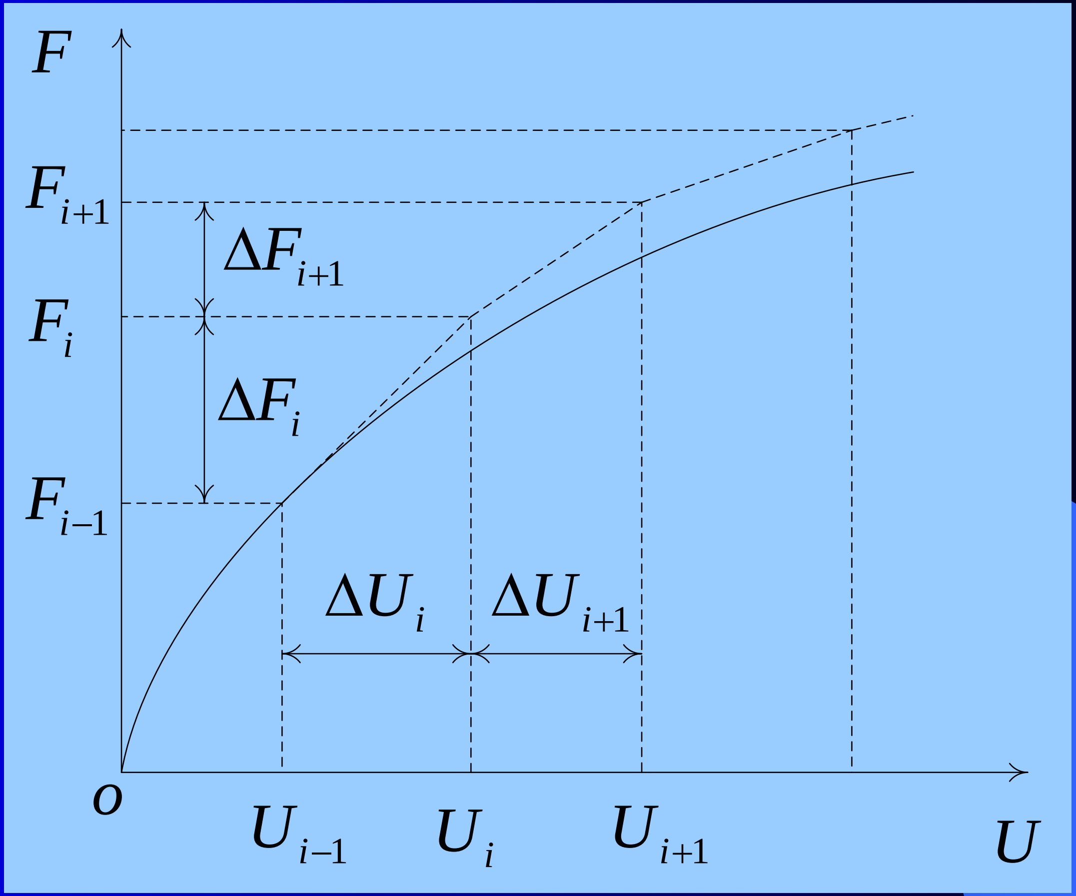


Simple Incremental Method

$$\mathbf{K}_{i-1}\Delta\mathbf{U}_i = \Delta\mathbf{F}_i$$

$$\Delta\mathbf{U}_i = \left(\mathbf{K}_{i-1}\right)^{-1}\Delta\mathbf{F}_i$$

$$\mathbf{U}_i = \mathbf{U}_{i-1} + \Delta\mathbf{U}_i$$



Newton-Raphson Incremental Method

$$\mathbf{K}_i^0 = \mathbf{K}_{i-1}$$

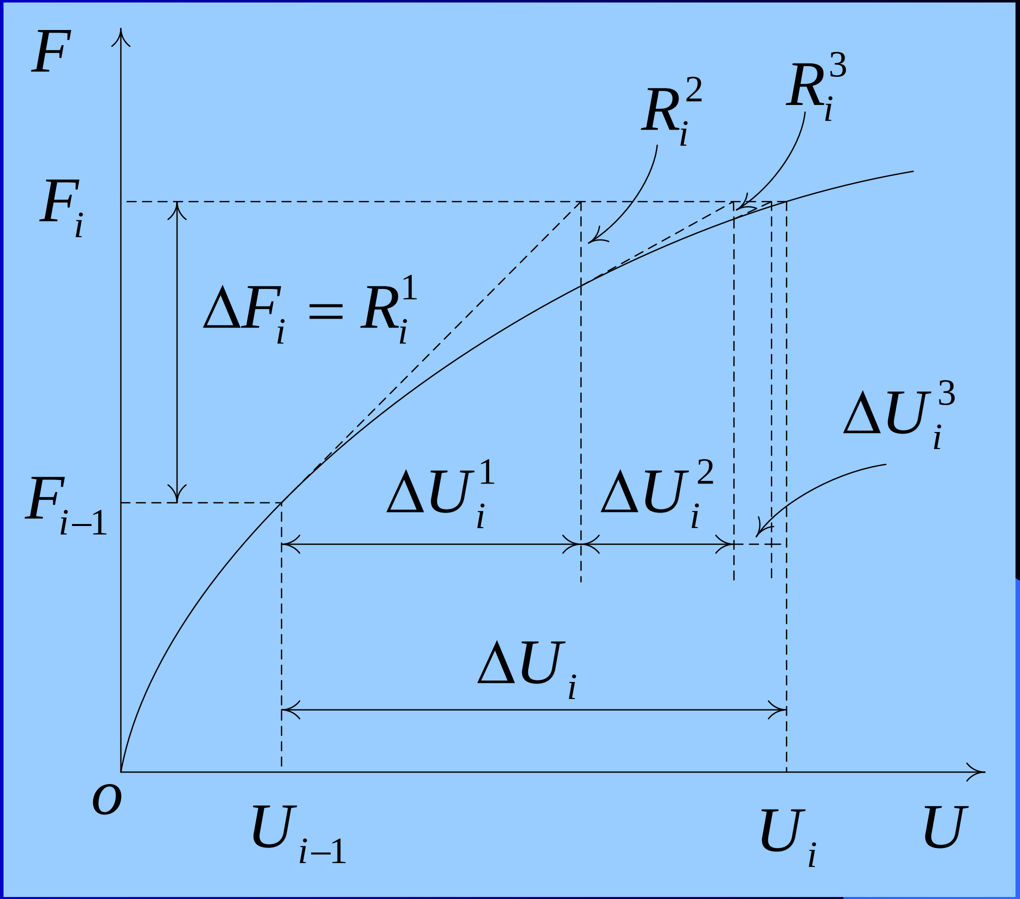
$\mathbf{R} =$ *Difference Between External and Internal Forces*

$$\mathbf{R}_i^1 = \Delta\mathbf{F}_i$$

$$\mathbf{K}_i^0 \Delta\mathbf{U}_i^1 = \mathbf{R}_i^1 \rightarrow \Delta\mathbf{U}_i^1 = \left(\mathbf{K}_i^0\right)^{-1}\mathbf{R}_i^1$$

$\mathbf{R}_i^2$ *is calculated*

$$\mathbf{K}_i^1 \Delta\mathbf{U}_i^2 = \mathbf{R}_i^2 \rightarrow \Delta\mathbf{U}_i^2 = \left(\mathbf{K}_i^1\right)^{-1}\mathbf{R}_i^2$$



If $\mathbf{R}_i^n$ *is negligible then* : $\Delta\mathbf{U}_i = \sum_{j=1}^{n-1} \Delta\mathbf{U}_i^j$

$$\mathbf{U}_i = \mathbf{U}_{i-1} + \Delta\mathbf{U}_i$$

Disadvantage of Newton-Raphson Incremental Method:

- A new stiffness matrix has to be formed and solved for each sub-step.

- At some cases, as non-associated plasticity, the stiffness matrix becomes non-symmetric and then non-symmetric solvers are required.

Modified Newton-Raphson Incremental Method

$\mathbf{K} = \mathbf{K}_{i-1}$

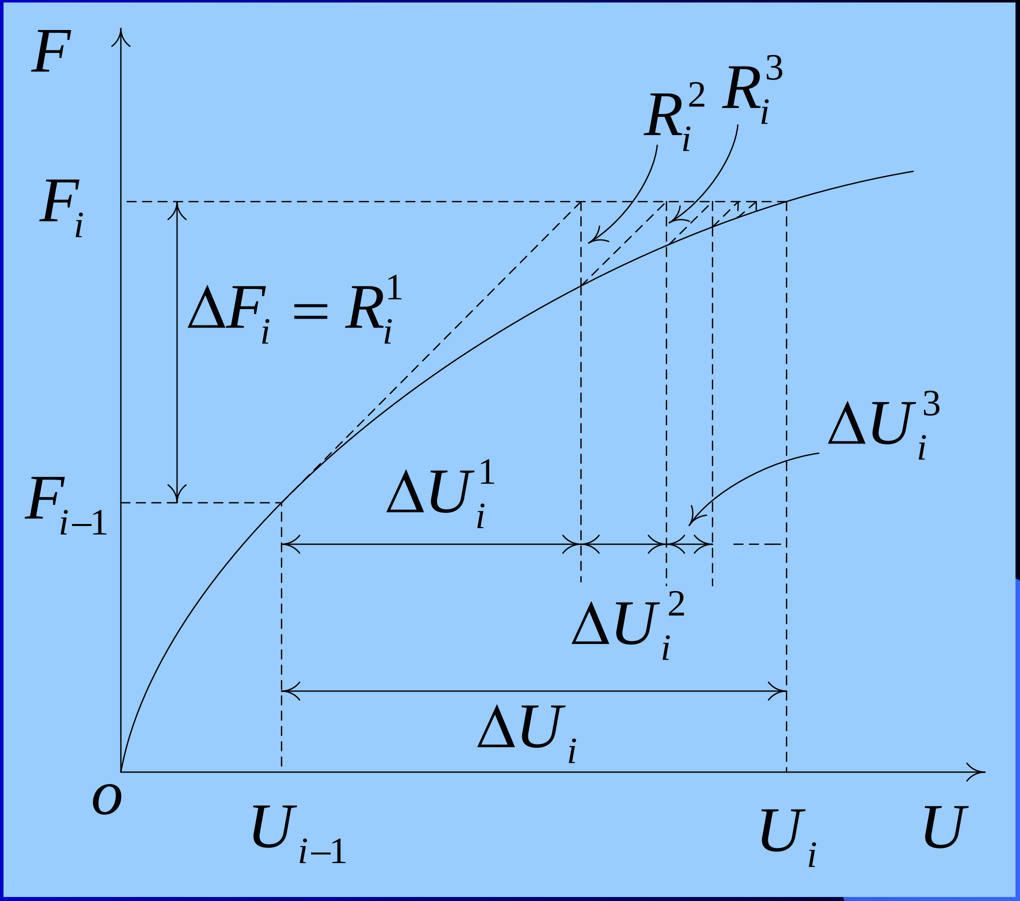
$\mathbf{R} = \textit{Difference Between External and Internal Forces}$

$\mathbf{R}_i^1 = \Delta \mathbf{F}_i$

$\mathbf{K}\Delta \mathbf{U}_i^1 = \mathbf{R}_i^1 \rightarrow \Delta \mathbf{U}_i^1 = \mathbf{K}^{-1}\mathbf{R}_i^1$

$\mathbf{R}_i^2$ *is calculated*

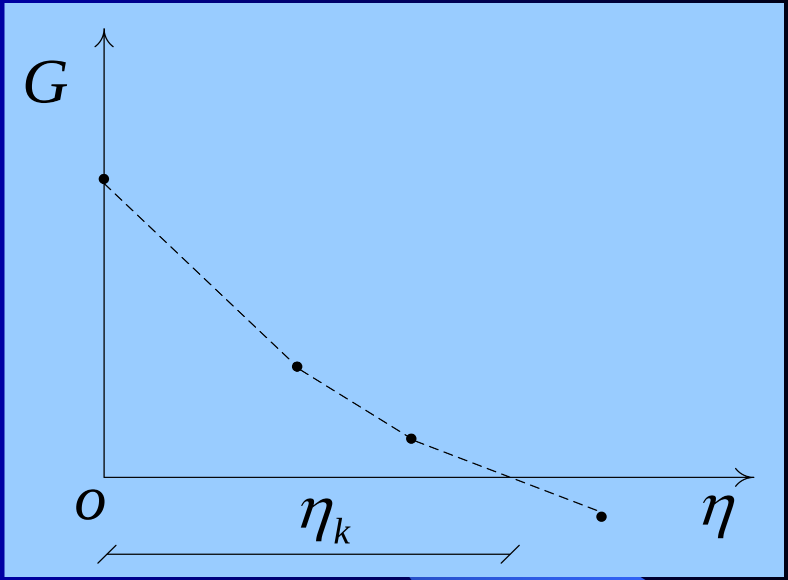
$\mathbf{K}\Delta \mathbf{U}_i^2 = \mathbf{R}_i^2 \rightarrow \Delta \mathbf{U}_i^2 = \mathbf{K}^{-1}\mathbf{R}_i^2$



If $\mathbf{R}_i^n$ *is negligible then* : $\Delta \mathbf{U}_i = \sum_{j=1}^{n-1} \Delta \mathbf{U}_i^j$

$\mathbf{U}_i = \mathbf{U}_{i-1} + \Delta \mathbf{U}_i$

Line Search Procedure for Acceleration of Convergence



$$\mathbf{K} = \mathbf{K}_{i-1}$$

$\mathbf{R} = $ *Difference Between External and Internal Forces*

$$\mathbf{R}_i^1 = \Delta\mathbf{F}_i$$

$$\mathbf{K}\Delta\mathbf{U}_i^1 = \mathbf{R}_i^1 \rightarrow \Delta\mathbf{U}_i^1 = (1+\eta_k)\mathbf{K}^{-1}\mathbf{R}_i^1; \eta_0 = 0$$

$\mathbf{R}_i^2$ *is calculated*

$$G = (\Delta\mathbf{U}_i^1)\mathbf{R}_i^2$$

η_k *is calculated so that* G *is made zero*

$$\mathbf{K}\Delta\mathbf{U}_i^2 = \mathbf{R}_i^2 \rightarrow \Delta\mathbf{U}_i^2 = \mathbf{K}^{-1}\mathbf{R}_i^2$$

If $\mathbf{R}_i^n$ *is negligible then* : $\Delta\mathbf{U}_i = \sum_{j=1}^{n-1} \Delta\mathbf{U}_i^j$

$$\mathbf{U}_i = \mathbf{U}_{i-1} + \Delta\mathbf{U}_i$$

Displacement Control Incremental Method

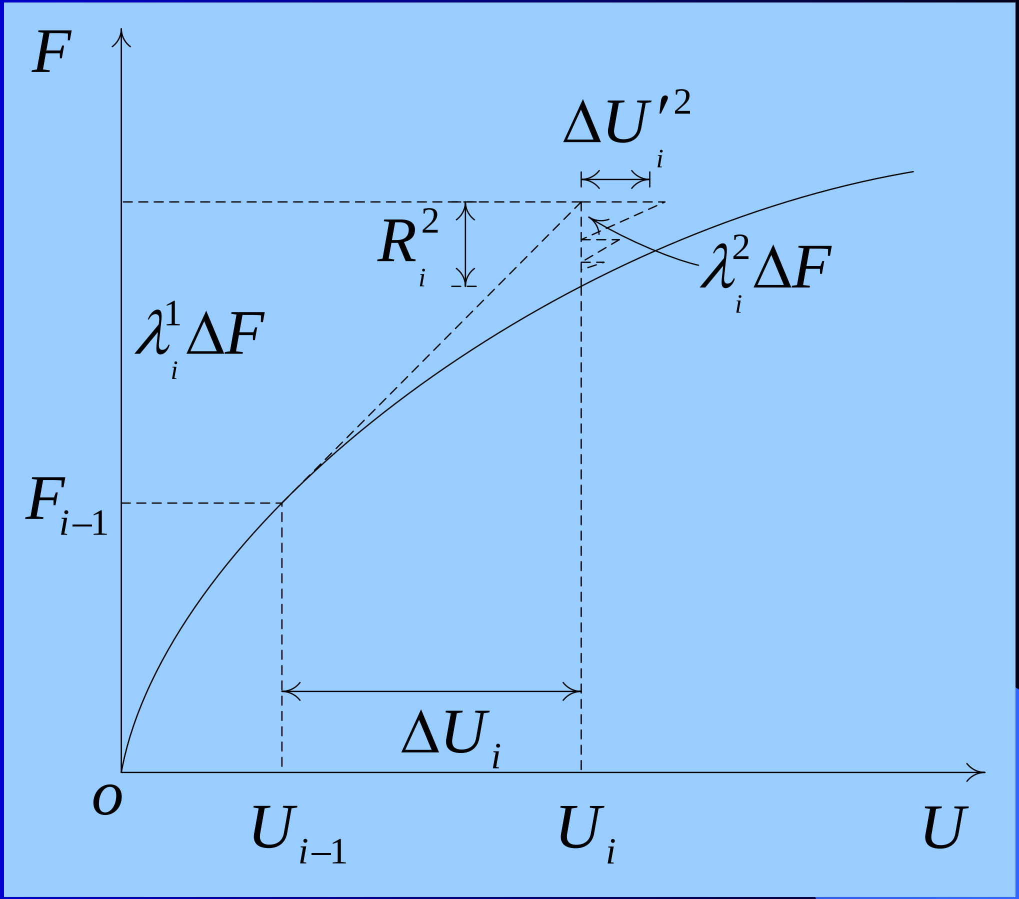
$$\mathbf{K}_i^0 = \mathbf{K}_{i-1}$$

$$\mathbf{R} = \textit{Difference Between External and Internal Forces}$$

$$\mathbf{R}_i^1 = \lambda_i^1 \Delta\mathbf{F}$$

$$\mathbf{K}_i^0 \Delta\mathbf{U}_i^1 = \mathbf{R}_i^1 = \lambda_i^1 \Delta\mathbf{F}$$

$$\therefore \Delta\mathbf{U}_i^1 = \lambda_i^1 \underbrace{\left(\mathbf{K}_i^0\right)^{-1} \Delta\mathbf{F}}_{\Delta\mathbf{U}_i''^1} = \lambda_i^1 \Delta\mathbf{U}_i''^1$$



$$\textit{If the } z-\textit{th degree of freedom} \left(\Delta U_{iz}\right) \textit{is prescribed},$$

$$\textit{then}: \Delta U_{iz} = \Delta U_{iz}^1 = \lambda_i^1 \Delta U_{iz}''^1 \rightarrow \lambda_i^1 = \frac{\Delta U_{iz}}{\Delta U_{iz}''^1}$$

Displacement Control Incremental Method (continue…)

$$\mathbf{R}_i^2 \textit{ is calculated}$$

$$\mathbf{K}_i^1 \Delta\mathbf{U}_i^2 = \mathbf{R}_i^2 + \lambda_i^2 \Delta\mathbf{F} \rightarrow \Delta\mathbf{U}_i^2 = \left(\mathbf{K}_i^1\right)^{-1}\left(\mathbf{R}_i^2 + \lambda_i^2 \Delta\mathbf{F}\right)$$

$$\Delta\mathbf{U}_i^2 = \underbrace{\left(\mathbf{K}_i^1\right)^{-1}\mathbf{R}_i^2}_{\Delta\mathbf{U}_i'^2} + \lambda_i^2 \underbrace{\left(\mathbf{K}_i^1\right)^{-1}\Delta\mathbf{F}}_{\Delta\mathbf{U}_i''^2} = \Delta\mathbf{U}_i'^2 + \lambda_i^2 \Delta\mathbf{U}_i''^2$$

$$\textit{Because of the } z-th \textit{ displacement}$$

$$\textit{was prescribed}: \Delta U_{iz}^2 = 0 \rightarrow \lambda_i^2 = -\frac{\Delta U_{iz}'^2}{\Delta U_{iz}''^2}$$

$$\textit{If } \mathbf{R}_i^n \textit{ is negligible then}: \begin{cases} \Delta\mathbf{U}_i = \sum_{j=1}^{n-1} \Delta\mathbf{U}_i^j \\ \Delta\mathbf{F}_i = \sum_{j=1}^{n-1} \lambda_i^j \Delta\mathbf{F} \end{cases}$$

Arc Length Control Method

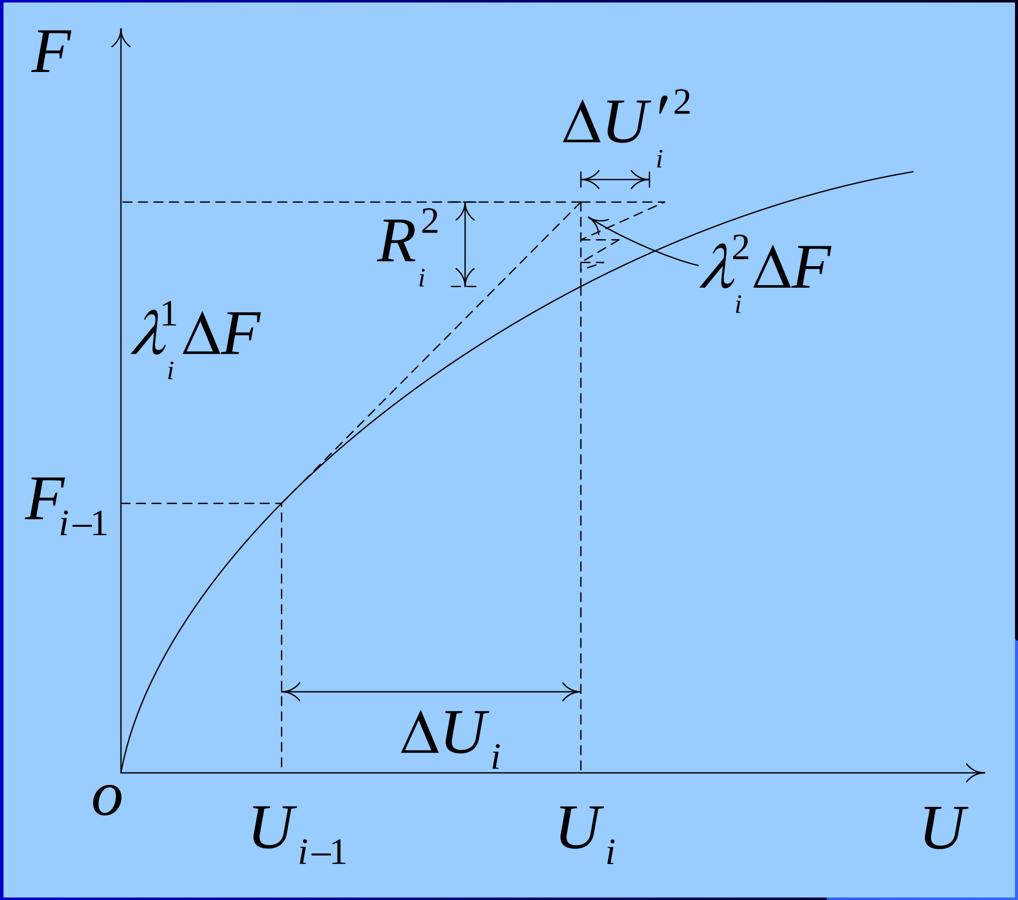
$$(\Delta \mathbf{U}_i)^T \Delta \mathbf{U}_i = \Delta l^2 \qquad \mathbf{K}_i^0 = \mathbf{K}_{i-1}$$

$\mathbf{R} =$ *Difference Between External and Internal Forces*

$$\mathbf{R}_i^1 = \lambda_i^1 \Delta \mathbf{F}$$

$$\mathbf{K}_i^0 \Delta \mathbf{U}_i^1 = \mathbf{R}_i^1 = \lambda_i^1 \Delta \mathbf{F}$$

$$\therefore \Delta \mathbf{U}_i^1 = \lambda_i^1 \underbrace{\left(\mathbf{K}_i^0\right)^{-1} \Delta \mathbf{F}}_{\Delta \mathbf{U}_i''^1} = \lambda_i^1 \Delta \mathbf{U}_i''^1$$



$$\left(\lambda_i^1 \Delta \mathbf{U}_i''^1\right)^T \left(\lambda_i^1 \Delta \mathbf{U}_i''^1\right) = \Delta l^2 \rightarrow \lambda_i^1 = \frac{\Delta l}{\sqrt{\left(\Delta \mathbf{U}_i''^1\right)^T \left(\Delta \mathbf{U}_i''^1\right)}}$$

Arc Length Control Method (continue…)

$\mathbf{R}_i^2$ *is calculated*

$$\mathbf{K}_i^1 \Delta\mathbf{U}_i^2 = \mathbf{R}_i^2 + \lambda_i^2 \Delta\mathbf{F} \rightarrow \Delta\mathbf{U}_i^2 = \left(\mathbf{K}_i^1\right)^{-1}\left(\mathbf{R}_i^2 + \lambda_i^2 \Delta\mathbf{F}\right)$$

$$\Delta\mathbf{U}_i^2 = \underbrace{\left(\mathbf{K}_i^1\right)^{-1}\mathbf{R}_i^2}_{\Delta\mathbf{U}_i'^2} + \lambda_i^2 \underbrace{\left(\mathbf{K}_i^1\right)^{-1}\Delta\mathbf{F}}_{\Delta\mathbf{U}_i''^2} = \Delta\mathbf{U}_i'^2 + \lambda_i^2 \Delta\mathbf{U}_i''^2$$

$$\Delta\mathbf{U}_i = \Delta\mathbf{U}_i^1 + \Delta\mathbf{U}_i^2 = \Delta\mathbf{U}_i^1 + \Delta\mathbf{U}_i'^2 + \lambda_i^2 \Delta\mathbf{U}_i''^2$$

$$\left(\Delta\mathbf{U}_i^1 + \Delta\mathbf{U}_i'^2 + \lambda_i^2 \Delta\mathbf{U}_i''^2\right)^T \left(\Delta\mathbf{U}_i^1 + \Delta\mathbf{U}_i'^2 + \lambda_i^2 \Delta\mathbf{U}_i''^2\right) = \Delta l^2$$

$$\left(\Delta\mathbf{U}_i''^2\right)^T \left(\Delta\mathbf{U}_i''^2\right)\left(\lambda_i^2\right)^2 + 2\left(\Delta\mathbf{U}_i^1 + \Delta\mathbf{U}_i'^2\right)^T \left(\Delta\mathbf{U}_i''^2\right)\left(\lambda_i^2\right) + \left(\Delta\mathbf{U}_i^1 + \Delta\mathbf{U}_i'^2\right)^T \left(\Delta\mathbf{U}_i^1 + \Delta\mathbf{U}_i'^2\right) - \Delta l^2 = 0$$

λ_i^2 *is calculated*

$$\textit{If } \mathbf{R}_i^n \textit{ is negligible then}: \begin{cases} \Delta\mathbf{U}_i = \sum_{j=1}^{n-1} \Delta\mathbf{U}_i^j \\ \Delta\mathbf{F}_i = \sum_{j=1}^{n-1} \lambda_i^j \Delta\mathbf{F} \end{cases}$$

Work Control Method

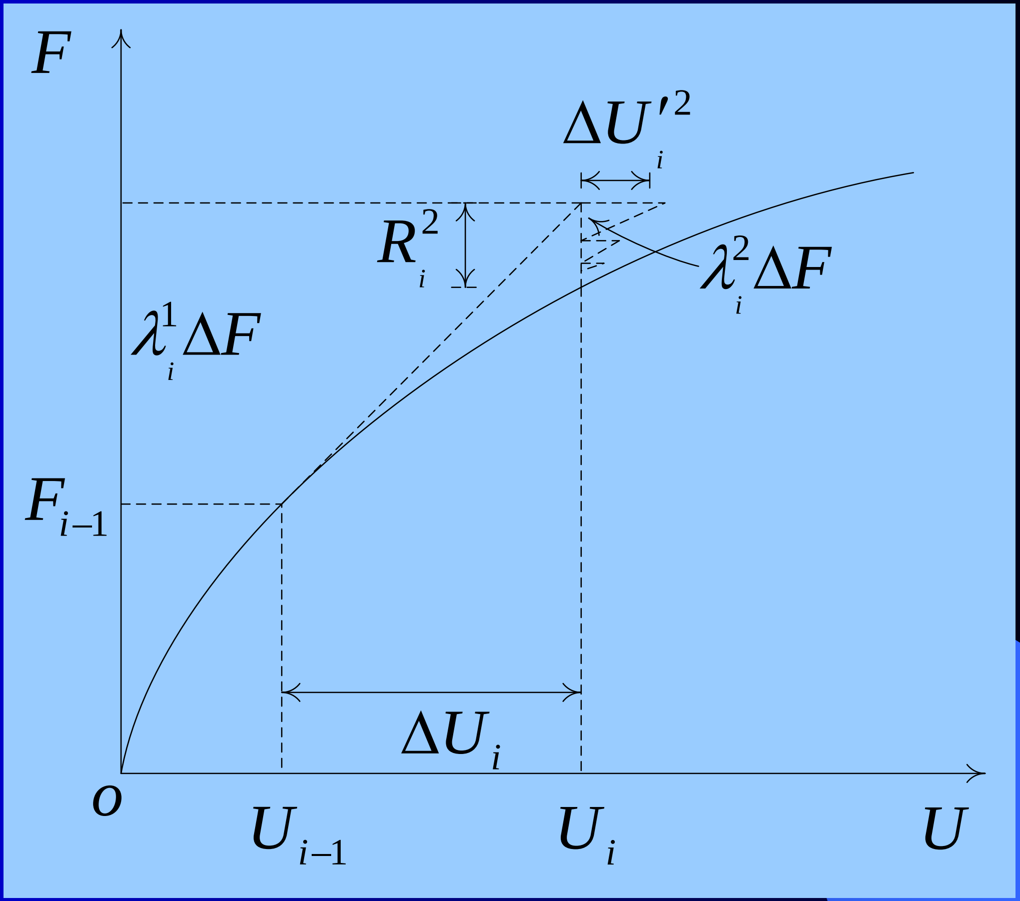
$$\left(\Delta\mathbf{U}_i\right)^T \Delta\mathbf{F}_i = \Delta W \qquad \mathbf{K}_i^0 = \mathbf{K}_{i-1}$$

$\mathbf{R}$ = *Difference Between External and Internal Forces*

$$\mathbf{R}_i^1 = \lambda_i^1 \Delta\mathbf{F}$$

$$\mathbf{K}_i^0 \Delta\mathbf{U}_i^1 = \mathbf{R}_i^1 = \lambda_i^1 \Delta\mathbf{F}$$

$$\therefore \Delta\mathbf{U}_i^1 = \lambda_i^1 \underbrace{\left(\mathbf{K}_i^0\right)^{-1} \Delta\mathbf{F}}_{\Delta\mathbf{U}_i''^1} = \lambda_i^1 \Delta\mathbf{U}_i''^1$$



$$\left(\lambda_i^1 \Delta\mathbf{U}_i''^1\right)^T \left(\lambda_i^1 \Delta\mathbf{F}\right) = \Delta W \rightarrow \lambda_i^1 = \sqrt{\frac{\Delta W}{\left(\Delta\mathbf{U}_i''^1\right)^T \left(\Delta\mathbf{F}\right)}}$$

Work Control Method (continue…)

$\mathbf{R}_i^2$ *is calculated*

$$\mathbf{K}_i^1 \Delta\mathbf{U}_i^2 = \mathbf{R}_i^2 + \lambda_i^2 \Delta\mathbf{F} \rightarrow \Delta\mathbf{U}_i^2 = \left(\mathbf{K}_i^1\right)^{-1}\left(\mathbf{R}_i^2 + \lambda_i^2 \Delta\mathbf{F}\right)$$

$$\Delta\mathbf{U}_i^2 = \underbrace{\left(\mathbf{K}_i^1\right)^{-1}\mathbf{R}_i^2}_{\Delta\mathbf{U}_i'^2} + \lambda_i^2 \underbrace{\left(\mathbf{K}_i^1\right)^{-1}\Delta\mathbf{F}}_{\Delta\mathbf{U}_i''^2} = \Delta\mathbf{U}_i'^2 + \lambda_i^2 \Delta\mathbf{U}_i''^2$$

$$\Delta\mathbf{U}_i = \Delta\mathbf{U}_i^1 + \Delta\mathbf{U}_i^2 = \Delta\mathbf{U}_i^1 + \Delta\mathbf{U}_i'^2 + \lambda_i^2 \Delta\mathbf{U}_i''^2$$

$$\Delta\mathbf{F}_i = \left(\lambda_i^1 + \lambda_i^2\right)\Delta\mathbf{F}$$

$$\left(\Delta\mathbf{U}_i^1 + \Delta\mathbf{U}_i'^2 + \lambda_i^2 \Delta\mathbf{U}_i''^2\right)^T \left(\lambda_i^1 + \lambda_i^2\right)\Delta\mathbf{F} = \Delta W$$

$$\left(\Delta\mathbf{U}_i''^2\right)^T (\Delta\mathbf{F})\left(\lambda_i^2\right)^2 + \left(\Delta\mathbf{U}_i^1 + \Delta\mathbf{U}_i'^2 + \lambda_i^1 \Delta\mathbf{U}_i''^2\right)^T (\Delta\mathbf{F})\left(\lambda_i^2\right) + \left(\lambda_i^1\right)\left(\Delta\mathbf{U}_i^1 + \Delta\mathbf{U}_i'^2\right)^T (\Delta\mathbf{F}) - \Delta W = 0$$

λ_i^2 *is calculated*

$$\textit{If } \mathbf{R}_i^n \textit{ is negligible then}: \begin{cases} \Delta\mathbf{U}_i = \sum_{j=1}^{n-1} \Delta\mathbf{U}_i^j \\ \Delta\mathbf{F}_i = \sum_{j=1}^{n-1} \lambda_i^j \Delta\mathbf{F} \end{cases}$$