Plastic Analysis and Design of Structures

Part 2

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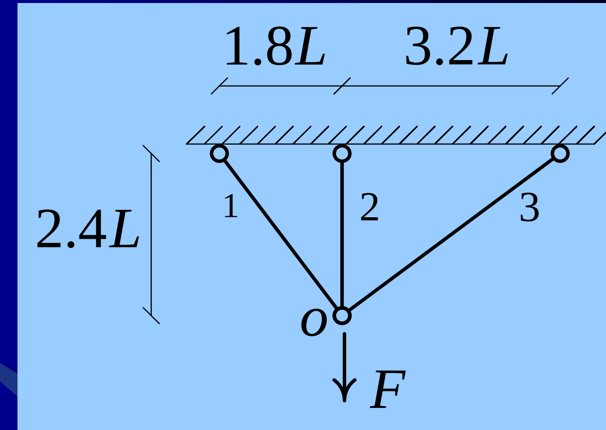
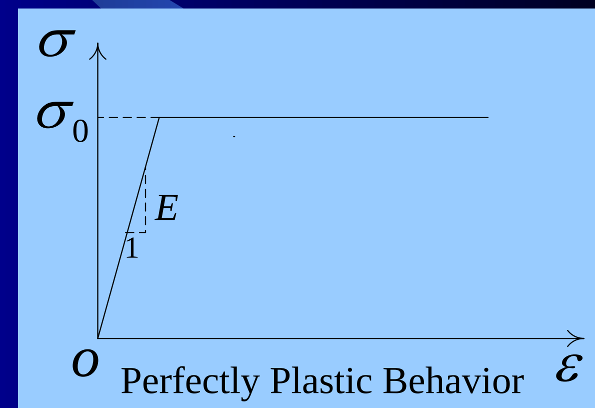
Limit Analysis,
Kinematic Approach, Example:



$$\begin{cases} A_1 = A \\ A_2 = A \\ A_3 = A \end{cases} \qquad \begin{cases} \sigma_{y1} = \sigma_0 \\ \sigma_{y2} = \sigma_0 \\ \sigma_{y3} = \sigma_0 \end{cases}$$

$$\sigma_0 A = S_0 \qquad \begin{cases} \vec{U}_{xo} = u \\ \downarrow U_{yo} = v \end{cases}$$



$$Kinematic\ Equations: \begin{cases} \dot{e}_1 = 0.6\dot{u} + 0.8\dot{v} \\ \dot{e}_2 = \dot{v} \\ \dot{e}_3 = -0.8\dot{u} + 0.6\dot{v} \end{cases}$$

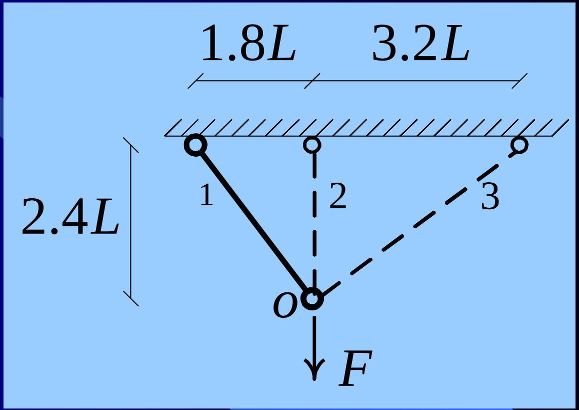
$$Equilibrium\ Equations: \begin{cases} \sum F_x = 0 \\ \sum F_y = F \end{cases} \rightarrow \begin{cases} 0.6S_1 - 0.8S_3 = 0 \\ 0.8S_1 + S_2 + 0.6S_3 = F \end{cases}$$

-The truss is statically indeterminate to the first degree and so collapse occurred if two bars yield, at least.

Limit Analysis, Kinematic Approach
Example, continue…:

-During collapse, the length of the elastic bars and the axial force of the Plastic bars remain constant.

-Alternative 1 using the equilibrium equations:
Bars 2 and 3 yield and bar 1 remains elastic.



$$\dot{e}_1 = 0.6\dot{u} + 0.8\dot{v} = 0 \quad \Rightarrow \quad \dot{u} = -1.333\dot{v}$$

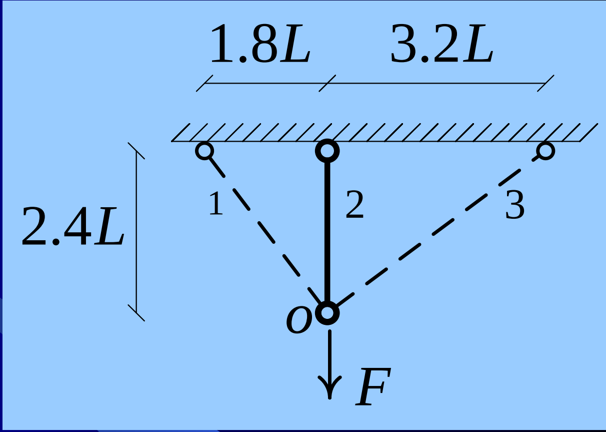
$$Kinematic\ Equations: \begin{cases} \dot{e}_1 = 0.6\dot{u} + 0.8\dot{v} = 0 \\ \dot{e}_2 = \dot{v} \\ \dot{e}_3 = -0.8\dot{u} + 0.6\dot{v} = 1.667\dot{v} \end{cases}$$

$$S_2 = S_3 = S_0$$

$$Equilibrium\ Equations: \begin{cases} 0.6S_1 - 0.8S_3 = 0 \\ 0.8S_1 + S_2 + 0.6S_3 = F \end{cases} \quad \Rightarrow \quad S_1 = 1.333S_0, \quad F = 2.667S_0$$

This mechanism is impossible.

Limit Analysis, Kinematic Approach
Example, continue…:



-Alternative 2 using the principle of virtual work:
Bars 1 and 3 yield and bar 2 remains elastic.

$$\delta e_2 = \delta v = 0 \quad \Rightarrow \quad \delta u = arbitrary$$

$$Kinematic\ Equations: \begin{cases} \delta e_1 = 0.6\delta u + 0.8\delta v = 0.6\delta u \\ \delta e_2 = 0 \\ \delta e_3 = -0.8\delta u + 0.6\delta v = -0.8\delta u \end{cases}$$

$$S_1 = S_3 = S_0$$

$$\delta W_{\text{int}} = \delta W_{ext} \quad \Rightarrow \quad S_1\delta e_1 + S_2\delta e_2 + S_3\delta e_3 = F\delta v$$

$$S_0(0.6\delta u) + S_2(0) + S_0(-0.8\delta u) = F(0) \quad \Rightarrow \quad F = \frac{-0.2S_0\delta u}{0} = \infty$$

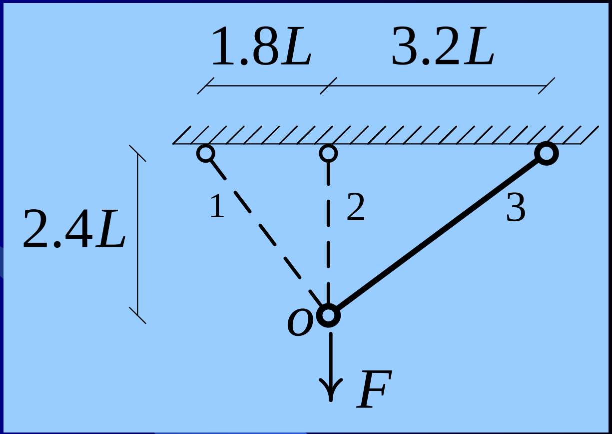
This mechanism is impossible.

Limit Analysis, Kinematic Approach

Example, continue…:

-Alternative 3 using the principle of energy conservation:

Bars 1 and 2 yield and bar 3 remains elastic.



$$\dot{e}_3 = -0.8\dot{u} + 0.6\dot{v} = 0 \;\Rightarrow\; \dot{u} = 0.75\dot{v}$$

$$Kinematic\ Equations : \begin{cases} \dot{e}_1 = 0.6\dot{u} + 0.8\dot{v} = 1.25\dot{v} \\ \dot{e}_2 = \dot{v} \\ \dot{e}_3 = -0.8\dot{u} + 0.6\dot{v} = 0 \end{cases}$$

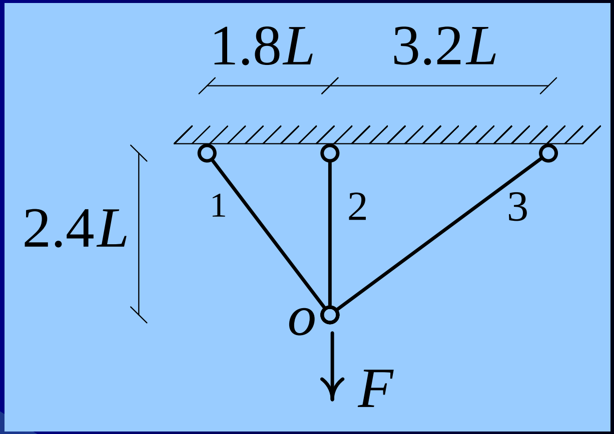
$$S_1 = S_2 = S_0$$

$$D_{\text{int}} = \dot{W}_{ext} \;\Rightarrow\; S_1\dot{e}_1 + S_2\dot{e}_2 + S_3\dot{e}_3 = F\dot{v}$$

$$S_0(1.25\dot{v}) + S_0(\dot{v}) + S_3(0) = F\dot{v} \;\Rightarrow\; F = 2.25S_0$$

$$0.6S_1 - 0.8S_3 = 0 \;\Rightarrow\; S_3 = 0.75S_0 \rightarrow$$ This is the true mechanism.

Limit Analysis, Kinematic Approach
Example, continue…:



-Alternative 1: $F = 2.667S_0$

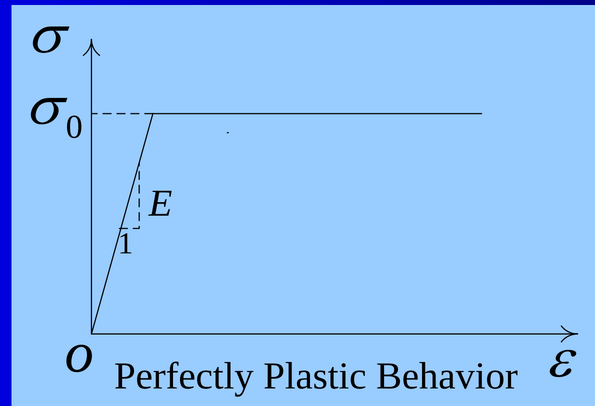
-Alternative 2: $F = \infty$

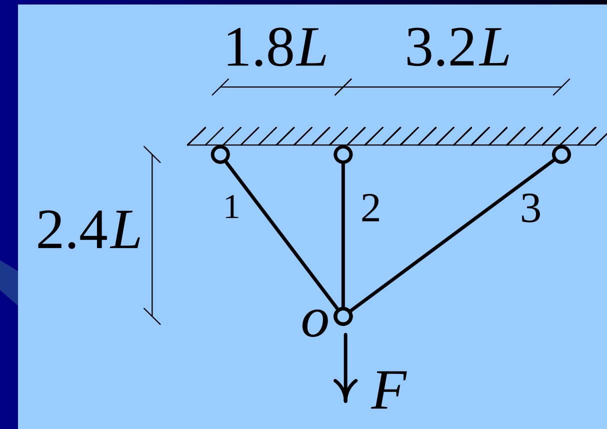
-Alternative 3: $F = 2.25S_0$ → This is the true mechanism.

-Upper bound theorem: All of the external forces resulted from any potential mechanisms are grater than or equal to actual collapse force.

Limit Analysis,
Static Approach, Example:

$$\begin{cases} A_1 = A \\ A_2 = A \\ A_3 = A \end{cases} \qquad \begin{cases} \sigma_{y1} = \sigma_0 \\ \sigma_{y2} = \sigma_0 \\ \sigma_{y3} = \sigma_0 \end{cases}$$





$$\sigma_0 A = S_0 \qquad \begin{cases} \vec{U}_{xo} = u \\ \downarrow U_{yo} = v \end{cases}$$

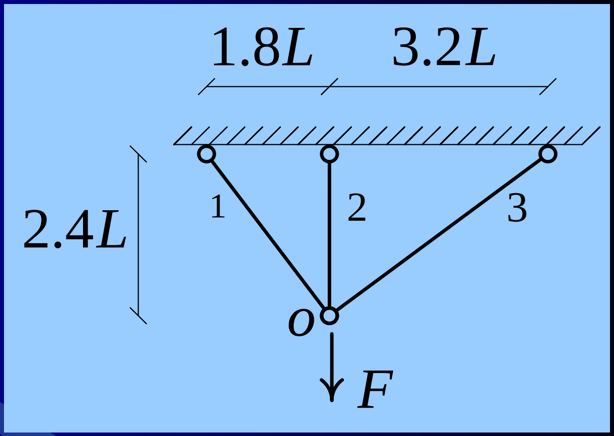
-The truss has two degree of freedom.

$$\text{Equilibrium Equations}: \begin{cases} \sum F_x = 0 \\ \sum F_y = F \end{cases} \rightarrow \begin{cases} 0.6S_1 - 0.8S_3 = 0 \\ 0.8S_1 + S_2 + 0.6S_3 = F \end{cases} \Rightarrow \begin{cases} S_3 = 0.75S_1 \\ S_2 = F - 1.25S_1 \end{cases}$$

-The condition of plastic admissibility.

$$\begin{cases} -S_0 \le S_1 \le S_0 \\ -S_0 \le S_2 \le S_0 \\ -S_0 \le S_3 \le S_0 \end{cases} \Rightarrow \begin{cases} -S_0 \le S_1 \le S_0 \\ -S_0 \le F - 1.25S_1 \le S_0 \\ -S_0 \le 0.75S_1 \le S_0 \end{cases} \Rightarrow \begin{cases} -S_0 \le S_1 \le S_0 \\ -S_0 \le F - 1.25S_1 \le S_0 \\ -1.333S_0 \le S_1 \le 1.333S_0 \end{cases}$$

Limit Analysis,
Static Approach, Example, continue…:



$$\begin{cases} -S_0 \le S_1 \le S_0 \\ -S_0 \le F - 1.25S_1 \le S_0 \\ -1.333S_0 \le S_1 \le 1.333S_0 \end{cases}$$

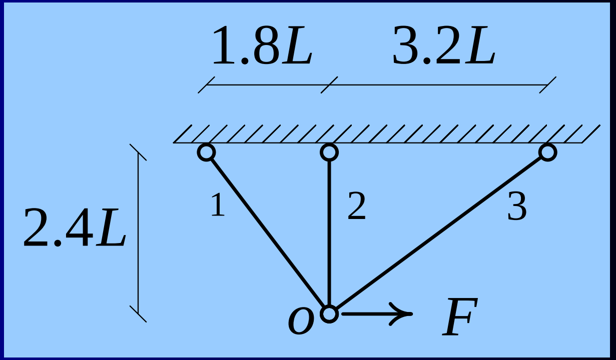
$$\begin{cases} -S_0 \le S_1 \le S_0 \\ -S_0 \le F - 1.25S_1 \le S_0 \end{cases} \Rightarrow \begin{cases} -S_0 \le S_1 \le S_0 \\ -S_0 + 1.25S_1 \le F \le S_0 + 1.25S_1 \end{cases}$$

$$-2.25S_0 \le F \le 2.25S_0 \Rightarrow F_0 = \pm 2.25S_0$$

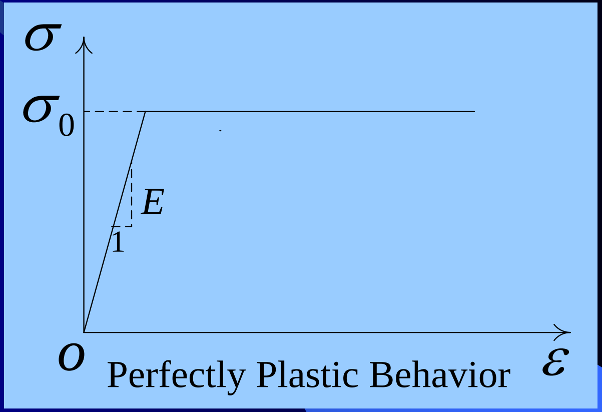
-Lower bound theorem: All of the external forces resulted from any statically plastic admissible internal forces, i.e. $-S_0 \le S_i \le S_0$,are lower than or equal to actual collapse force.

Limit Analysis, Kinematic Approach

Example:



$$\begin{cases} A_1 = 0.90A \\ A_2 = 0.96A \\ A_3 = A \end{cases} \qquad \begin{cases} \sigma_{y1} = 0.8\sigma_0 \\ \sigma_{y2} = 0.06\sigma_0 \\ \sigma_{y3} = \sigma_0 \end{cases} \qquad \sigma_0 A = S_0$$



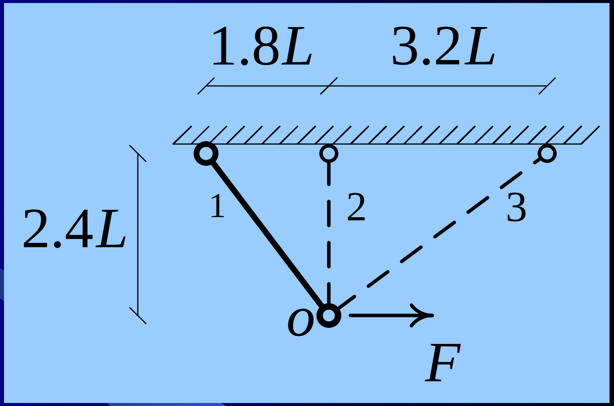
$$\begin{cases} \vec{U}_{xo} = u \\ \downarrow U_{yo} = v \end{cases} \qquad \begin{cases} S_{01} = 0.72S_0 \\ S_{02} = 0.0576S_0 \\ S_{03} = S_0 \end{cases}$$

$$Kinematic\ Equations: \begin{cases} \dot{e}_1 = 0.6\dot{u} + 0.8\dot{v} \\ \dot{e}_2 = \dot{v} \\ \dot{e}_3 = -0.8\dot{u} + 0.6\dot{v} \end{cases}$$

-The truss is statically indeterminate to first degree and so collapse occurred if two bars yield, at least.

-Number of potential mechanisms: $\binom{3}{2} = \dfrac{(3!)}{(2!)(1!)} = 3$

Limit Analysis, Kinematic Approach
Example, continue…:



-Alternative 1: Bars 2 and 3 have yield, but bar 1 is elastic.

$$\dot{e}_1 = 0.6\dot{u} + 0.8\dot{v} = 0 \quad \Rightarrow \quad \dot{v} = -0.75\dot{u}$$

$$Kinematic\ Equations: \begin{cases} \dot{e}_1 = 0.6\dot{u} + 0.8\dot{v} = 0 \\ \dot{e}_2 = \dot{v} = -0.75\dot{u} \\ \dot{e}_3 = -0.8\dot{u} + 0.6\dot{v} = -1.25\dot{u} \end{cases}$$

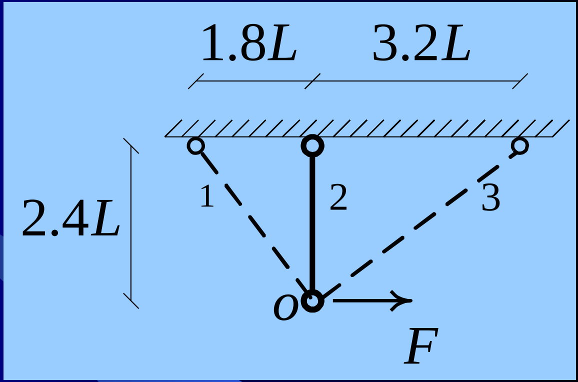
$$\begin{cases} S_2 = 0.0576 S_0 \\ S_3 = S_0 \end{cases}$$

$$D_{\text{int}} = \dot{W}_{ext} \quad \Rightarrow \quad S_1\dot{e}_1 + S_2\dot{e}_2 + S_3\dot{e}_3 = F\dot{u}$$

$$S_1(0) + 0.0576 S_0 (0.75\dot{u}) + S_0(1.25\dot{u}) = F\dot{u} \quad \Rightarrow \quad F = 1.2932 S_0$$

Limit Analysis, Kinematic Approach
Example, continue…:

-Alternative 2: Bars 1 and 3 have yield, but bar 2 is elastic.



$$\dot{e}_2 = \dot{v} = 0 \quad \Rightarrow \quad \dot{v} = 0$$

$$Kinematic\ Equations: \begin{cases} \dot{e}_1 = 0.6\dot{u} + 0.8\dot{v} = 0.6\dot{u} \\ \dot{e}_2 = \dot{v} = 0 \\ \dot{e}_3 = -0.8\dot{u} + 0.6\dot{v} = -0.8\dot{u} \end{cases}$$

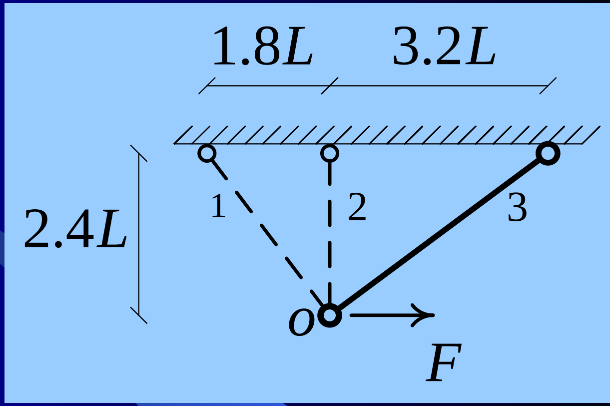
$$\begin{cases} S_1 = 0.72 S_0 \\ S_3 = S_0 \end{cases}$$

$$D_{\text{int}} = \dot{W}_{ext} \quad \Rightarrow \quad S_1\dot{e}_1 + S_2\dot{e}_2 + S_3\dot{e}_3 = F\dot{u}$$

$$0.72S_0(0.6\dot{u}) + S_2(0) + S_0(0.8\dot{u}) = F\dot{u} \quad \Rightarrow \quad F = 1.232 S_0$$

Limit Analysis, Kinematic Approach
Example, continue…:

-Alternative 3: bars 1 and 2 have yield, but bar 3 is elastic.



$$\dot{e}_3 = -0.8\dot{u} + 0.6\dot{v} = 0 \quad \Rightarrow \quad \dot{v} = 1.333\dot{u}$$

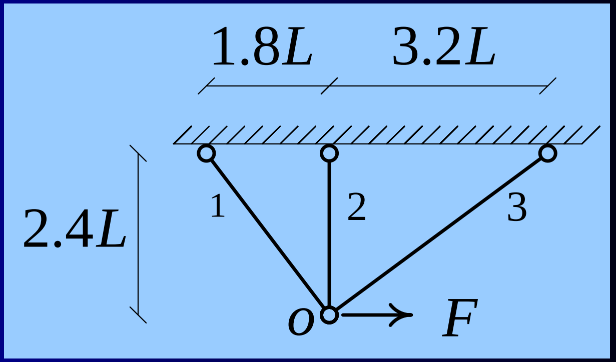
$$Kinematic\ Equations: \begin{cases} \dot{e}_1 = 0.6\dot{u} + 0.8\dot{v} = 1.667\dot{u} \\ \dot{e}_2 = \dot{v} = 1.333\dot{u} \\ \dot{e}_3 = -0.8\dot{u} + 0.6\dot{v} = 0 \end{cases}$$

$$\begin{cases} S_1 = 0.72 S_0 \\ S_2 = 0.0576 S_0 \end{cases}$$

$$D_{\text{int}} = \dot{W}_{ext} \quad \Rightarrow \quad S_1\dot{e}_1 + S_2\dot{e}_2 + S_3\dot{e}_3 = F\dot{u}$$

$$0.72S_0(1.667\dot{u}) + 0.0576S_0(1.333\dot{u}) + S_3(0) = F\dot{u} \quad \Rightarrow \quad F = 1.2768 S_0$$

Limit Analysis, Kinematic Approach
Example, continue…:



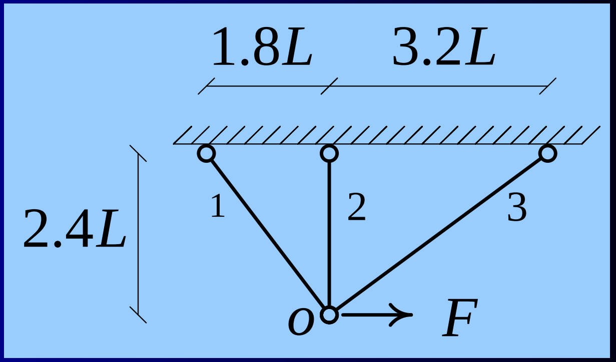
-Alternative 1: $F = 1.2932 S_0$

-Alternative 2: $F = 1.232$ → This is the true mechanism.

-Alternative 3: $F = 1.2768 S_0$

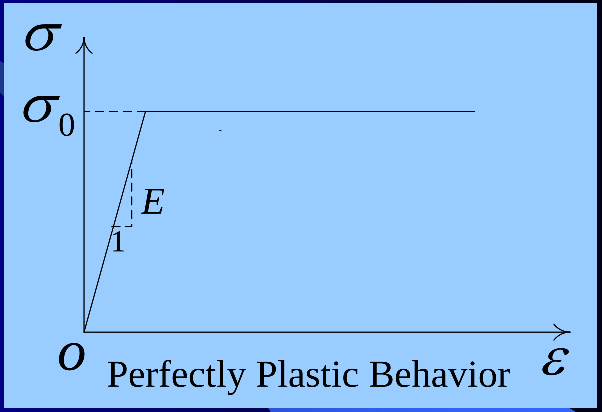
Limit Analysis, Static Approach

Example:



$$\begin{cases} A_1 = 0.90A \\ A_2 = 0.96A \\ A_3 = A \end{cases} \qquad \begin{cases} \sigma_{y1} = 0.8\sigma_0 \\ \sigma_{y2} = 0.06\sigma_0 \\ \sigma_{y3} = \sigma_0 \end{cases}$$

$$\sigma_0 A = S_0 \qquad \begin{cases} S_{01} = 0.72S_0 \\ S_{02} = 0.0576S_0 \\ S_{03} = S_0 \end{cases}$$



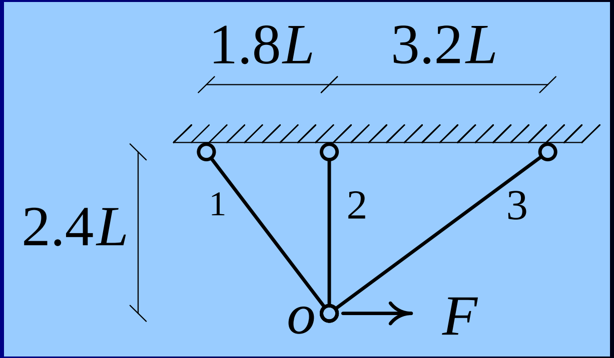
-The truss has two degree of freedom.

$$Equilibrium\ Equations: \begin{cases} \sum F_x = F \\ \sum F_y = 0 \end{cases} \rightarrow \begin{cases} 0.6S_1 - 0.8S_3 = F \\ 0.8S_1 + S_2 + 0.6S_3 = 0 \end{cases} \qquad \begin{cases} S_3 = 0.75S_1 - 1.25F \\ S_2 = -1.25S_1 + 0.75F \end{cases}$$

-The condition of plastic admissibility.

$$\begin{cases} -0.72S_0 \le S_1 \le 0.72S_0 \\ -0.0576S_0 \le S_2 \le 0.0576S_0 \\ -S_0 \le S_3 \le S_0 \end{cases} \Rightarrow \begin{cases} -0.72S_0 \le S_1 \le 0.72S_0 \\ -0.0576S_0 \le -1.25S_1 + 0.75F \le 0.0576S_0 \\ -S_0 \le 0.75S_1 - 1.25F \le S_0 \end{cases}$$

Limit Analysis, Static Approach
Example, continue…:



$$\begin{cases} -0.72S_0 \le S_1 \le 0.72S_0 \\ -0.0576S_0 \le -1.25S_1 + 0.75F \le 0.0576S_0 \\ -S_0 \le 0.75S_1 - 1.25F \le S_0 \end{cases}$$

$$\begin{cases} -0.72S_0 \le S_1 \le 0.72S_0 \\ -0.0768S_0 + 1.667S_1 \le F \le 0.0768S_0 + 1.667S_1 \\ -0.8S_0 + 0.6S_1 \le F \le 0.8S_0 + 0.6S_1 \end{cases}$$

$$\begin{cases} -1.2768S_0 \le F \le 1.2768S_0 \\ -1.232S_0 \le F \le 1.232S_0 \end{cases}$$

$$-1.232S_0 \le F \le 1.232S_0$$

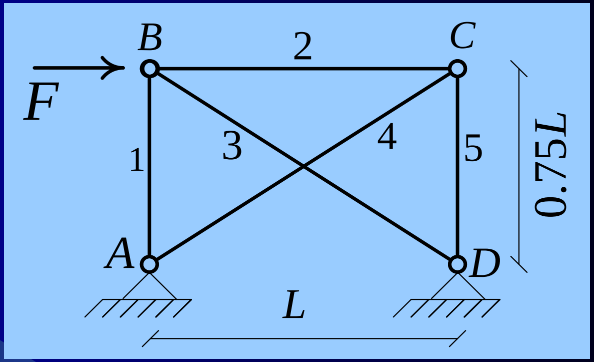
$$F_0 = \pm 1.232$$

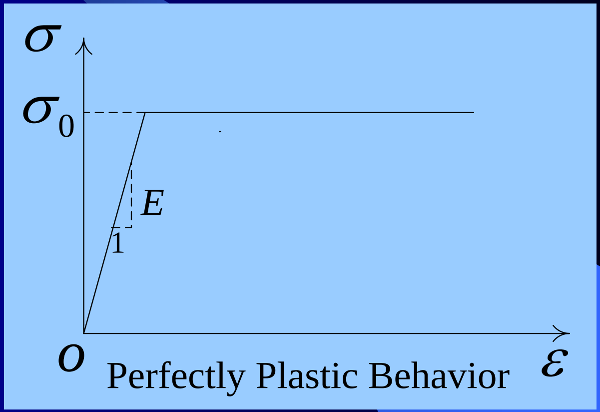
Limit Analysis,
Kinematic Approach, Example:

$$A_1 = A_2 = A_3 = A_4 = A_5 = A$$

$$\sigma_{y1} = \sigma_{y2} = \sigma_{y3} = \sigma_{y4} = \sigma_{y5} = \sigma_0$$

$$\sigma_0 A = S_0$$





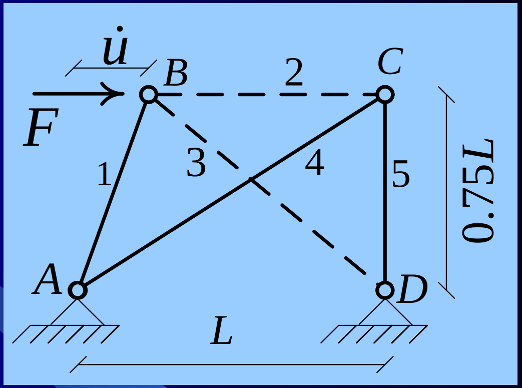
-The truss is statically indeterminate to the first degree and so collapse occurred if two bars yield, at least.

-Number of potential mechanisms:

$$\binom{5}{2} = \frac{(5!)}{(2!)(3!)} = 10$$

Limit Analysis,
Kinematic Approach, Example, continue…:

-One of the kinematically admissible mechanisms having positive external power:



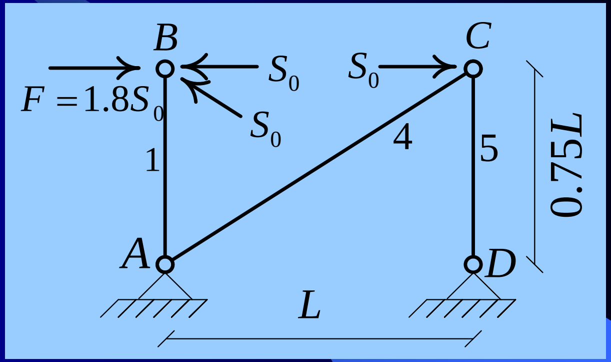
$$Kinematic\ Equations: \begin{cases} \dot{e}_1 = 0 \\ \dot{e}_2 = -\dot{u} \\ \dot{e}_3 = -0.8\dot{u} \\ \dot{e}_4 = 0 \\ \dot{e}_5 = 0 \end{cases}$$

$$D_{\text{int}} = \dot{W}_{ext} \Longrightarrow S_1\dot{e}_1 + S_2\dot{e}_2 + S_3\dot{e}_3 + S_4\dot{e}_4 + S_5\dot{e}_5 = F_k\dot{u}$$

$$S_0(\dot{u}) + S_0(0.8\dot{u}) = F_k\dot{u} \Longrightarrow F_k = 1.8S_0 \Longrightarrow F_0 \leq 1.8S_0$$

Limit Analysis,
Kinematic Approach, Example, continue…:

$$Equilibrium\ Equations: \begin{cases} \sum F_{yB} = 0 \\ \sum F_{xC} = 0 \\ \sum F_{yC} = 0 \end{cases} \rightarrow \begin{cases} S_1 - 0.6S_0 = 0 \\ S_0 - 0.8S_4 = 0 \\ 0.6S_4 + S_5 = 0 \end{cases}$$



$$\begin{cases} S_1 = 0.6S_0 \\ S_4 = 1.25S_0 \\ S_5 = -0.75S_0 \end{cases}$$

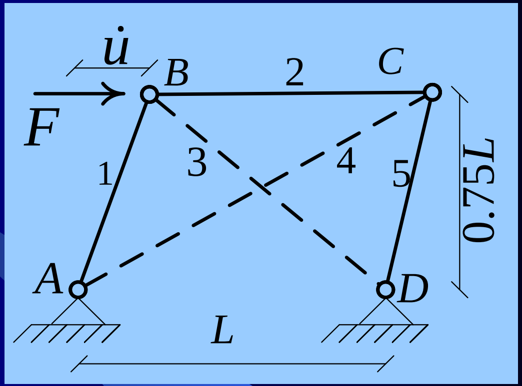
This doesn't comply with plastically admissible condition.

$$F_1 = \frac{1.8}{1.25} S_0 = 1.44S_0$$

$$1.44S_0 \le F_0 \le 1.8S_0$$

Limit Analysis,
Kinematic Approach, Example, continue…:

-Another kinematically admissible mechanisms having positive external power:



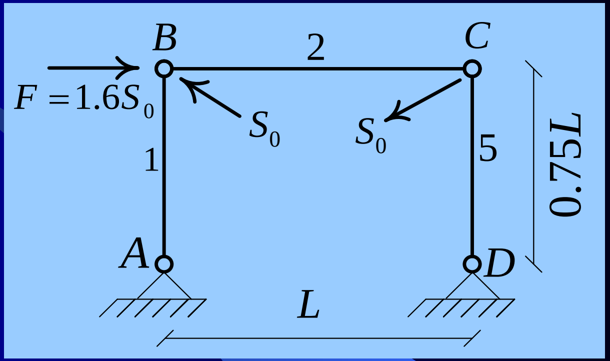
$$Kinematic\ Equations: \begin{cases} \dot{e}_1 = 0 \\ \dot{e}_2 = 0 \\ \dot{e}_3 = -0.8\dot{u} \\ \dot{e}_4 = 0.8\dot{u} \\ \dot{e}_5 = 0 \end{cases}$$

$$D_{\text{int}} = \dot{W}_{ext} \quad \Longrightarrow \quad S_1\dot{e}_1 + S_2\dot{e}_2 + S_3\dot{e}_3 + S_4\dot{e}_4 + S_5\dot{e}_5 = F_k\dot{u}$$

$$S_0(0.8\dot{u}) + S_0(0.8\dot{u}) = F_k\dot{u} \quad \Longrightarrow \quad F_k = 1.6S_0 \quad \Longrightarrow \quad 1.44S_0 \le F_0 \le 1.6S_0$$

Limit Analysis,
Kinematic Approach, Example, continue…:

$$\text{Equilibrium Equations}:\begin{cases}\sum F_{yB}=0\\ \sum F_{xC}=0\\ \sum F_{yC}=0\end{cases}\rightarrow\begin{cases}S_1-0.6S_0=0\\ S_2+0.8S_0=0\\ 0.6S_0+S_5=0\end{cases}$$



$$\begin{cases}S_1=0.6S_0\\ S_2=-0.8S_0\\ S_5=-0.6S_0\end{cases}$$

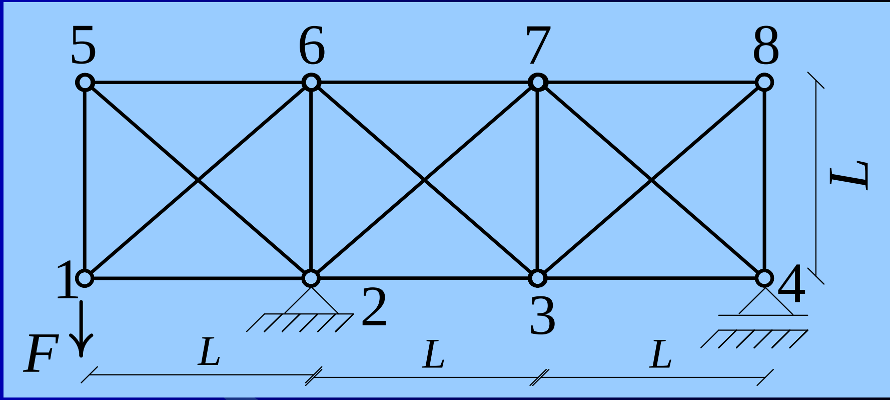
This comply with plastically admissible condition.

$$F_0=1.6S_0$$

Limit Analysis,
Kinematic Approach, Example:

Diagonal bars: ‘d’

The other bars: ‘b’

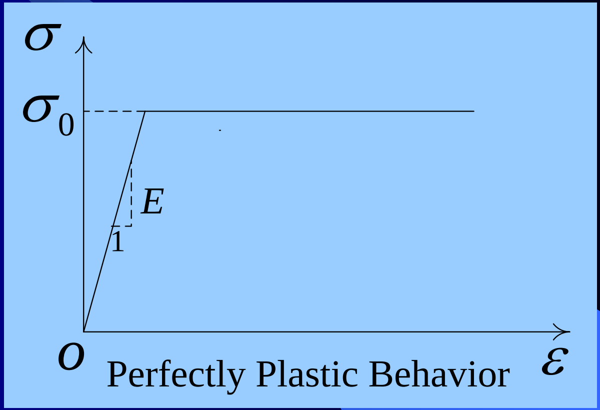


$$A_b = 2A_d = 2A$$

$$\sigma_{yd} = \sigma_{yb} = \sigma_0$$

$$S_{0d} = \sigma_0 A = S_0$$

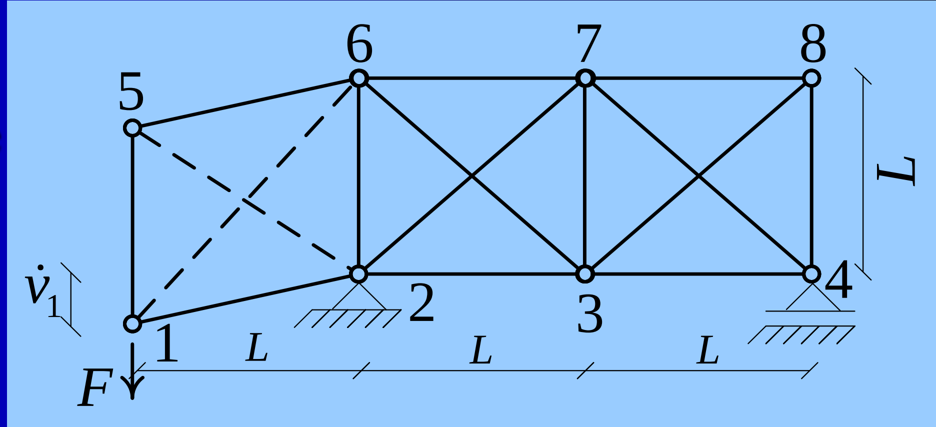
$$S_{0b} = \sigma_0 2A = 2S_0$$



-The truss is three degree statically indeterminate, but partial collapse can occurred if two bars yield.

Limit Analysis, Kinematic Approach, Example, continue…:

-One of the kinematically admissible mechanisms having positive external power :

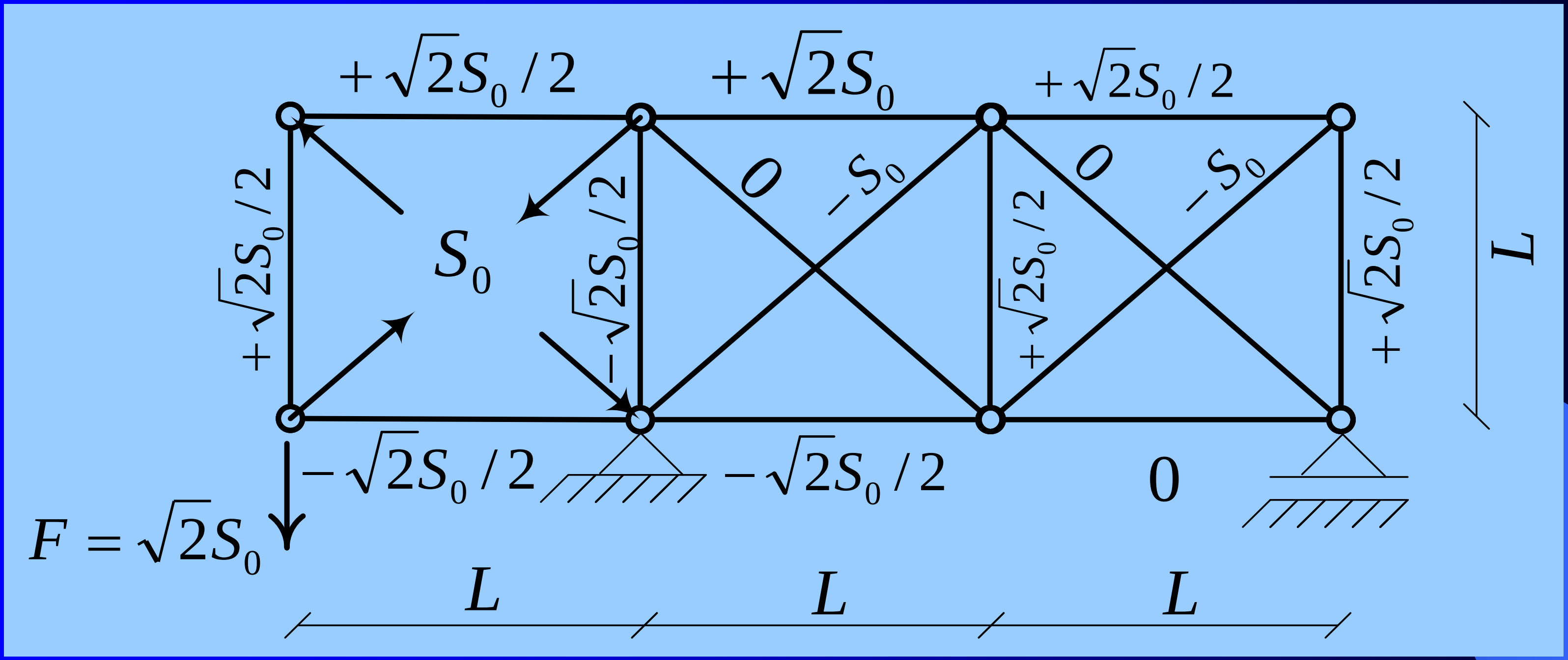


$$Kinematic\ Equations: \begin{cases} \dot{e}_{25} = -\dfrac{\dot{v}_1}{\sqrt{2}} \\ \dot{e}_{16} = +\dfrac{\dot{v}_1}{\sqrt{2}} \end{cases}$$

$$D_{\text{int}} = \dot{W}_{ext} \Longrightarrow S_{0d}\left|\dot{e}_{25}\right| + S_{0d}\left|\dot{e}_{16}\right| = F_k \dot{v}_1$$

$$S_0\left(\frac{\dot{v}_1}{\sqrt{2}}\right) + S_0\left(\frac{\dot{v}_1}{\sqrt{2}}\right) = F_k \dot{v}_1 \Longrightarrow F_k = \sqrt{2} S_0 \Longrightarrow F_0 \le \sqrt{2} S_0$$

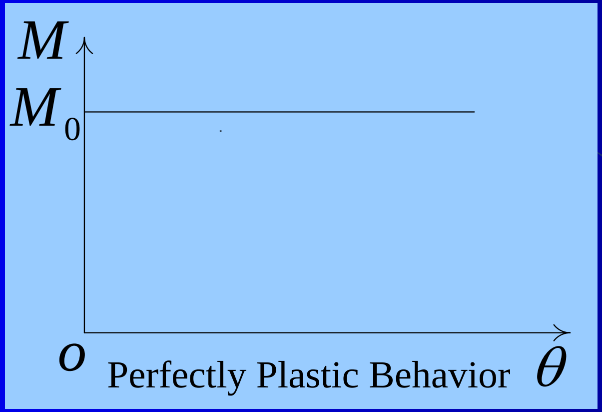
Limit Analysis, Kinematic Approach, Example, continue…:

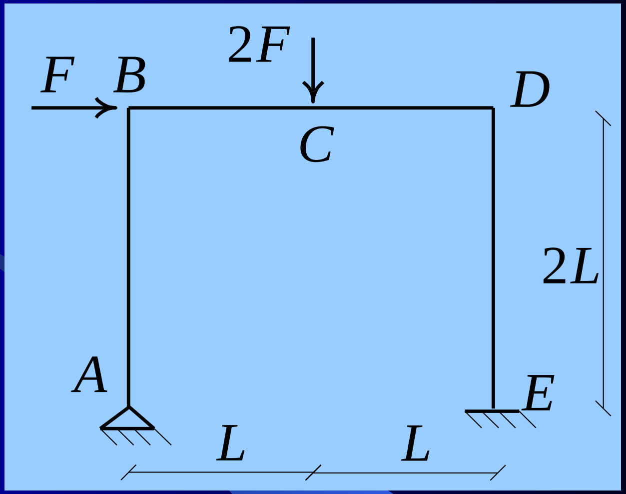


These axial forces comply with plastically admissible condition.

$$\boxed{F_0=\sqrt{2}S_0}$$

Limit Analysis in Frames,
Kinematic Approach, Example:





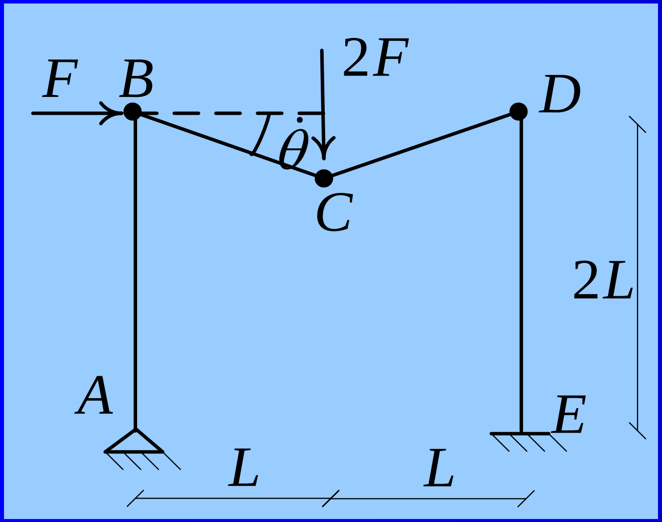
-The frame is statically indeterminate to the second degree and has four critical sections as B, C, D and E, so collapse occurred if the three hinges are inserted in these sections.

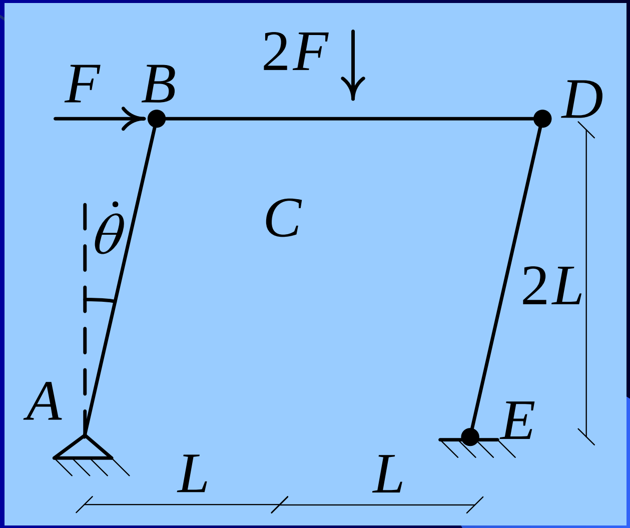
-Number of potential mechanisms:

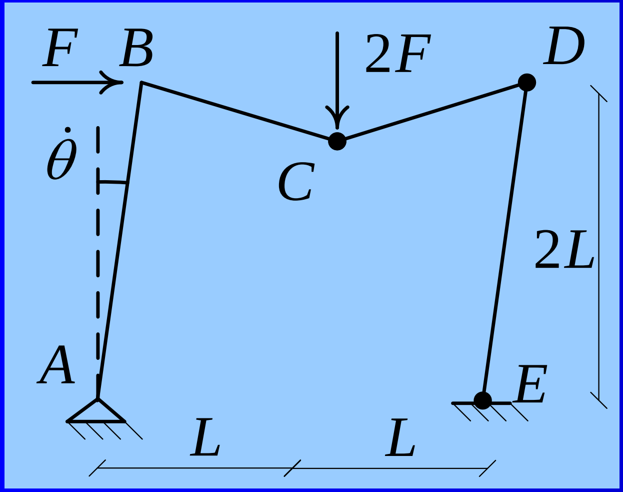
$$\binom{4}{3} = \frac{(4!)}{(3!)(1!)} = 4$$

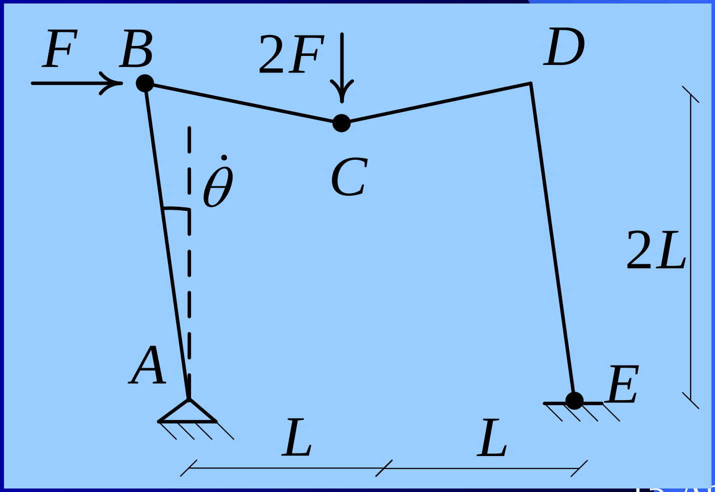
Limit Analysis in Frames, Kinematic Approach, Example, continue…:

-The potential mechanisms:









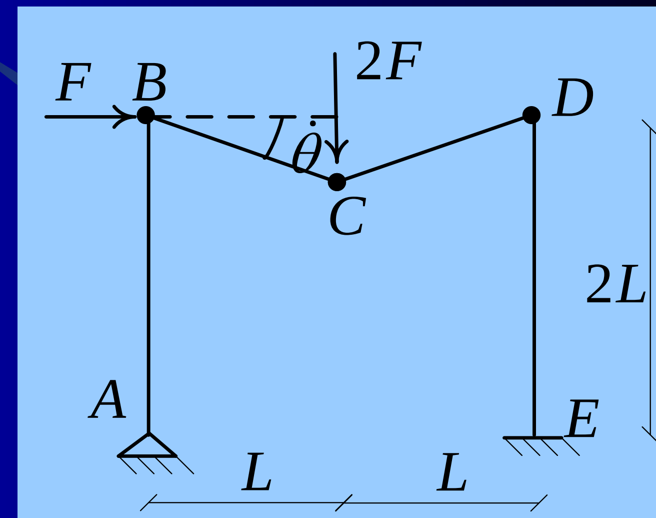
Limit Analysis in Frames,
Kinematic Approach, Example, continue…:

-Alternative 1:

$$D_{\text{int}} = \dot{W}_{ext}$$

$$M_0(\dot{\theta}) + M_0(2\dot{\theta}) + M_0(\dot{\theta}) = 2F(\dot{\theta}L)$$

$$F = \frac{2M_0}{L}$$

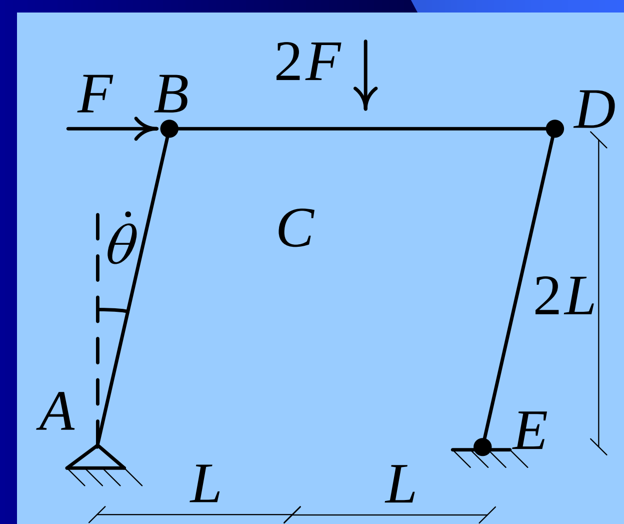


-Alternative 2:

$$D_{\text{int}} = \dot{W}_{ext}$$

$$M_0(\dot{\theta}) + M_0(\dot{\theta}) + M_0(\dot{\theta}) = F(\dot{\theta}2L)$$

$$F = \frac{1.5M_0}{L}$$



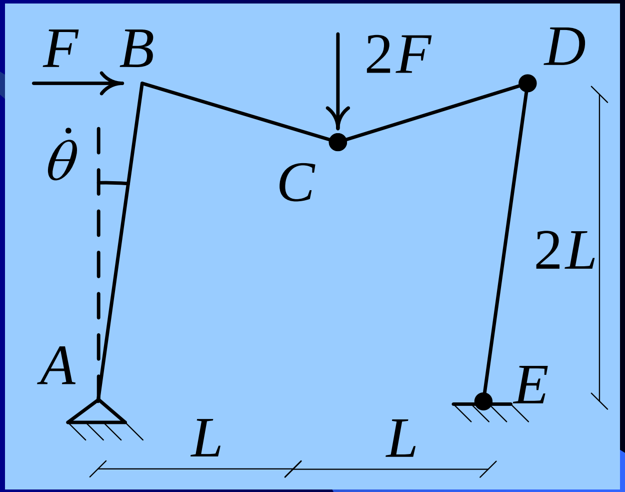
Limit Analysis in Frames, Kinematic Approach, Example, continue…:

-Alternative 3:

$$D_{\text{int}} = \dot{W}_{ext}$$

$$M_0(2\dot{\theta}) + M_0(2\dot{\theta}) + M_0(\dot{\theta}) = F(\dot{\theta}2L) + 2F(\dot{\theta}L)$$

$$F = \frac{1.25M_0}{L}$$

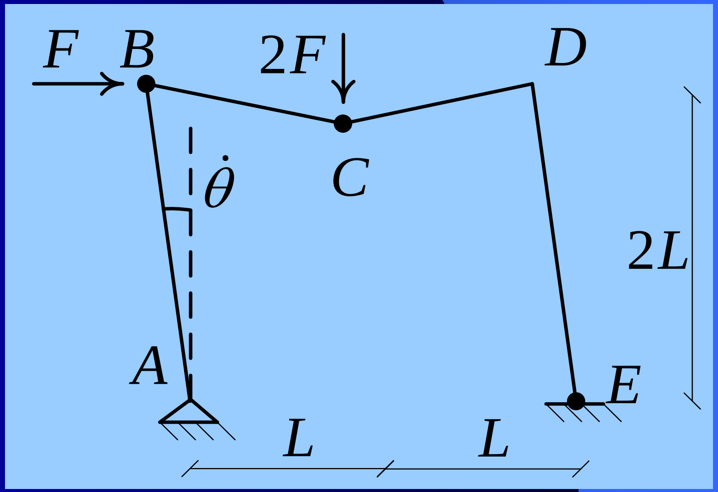


-Alternative 4:

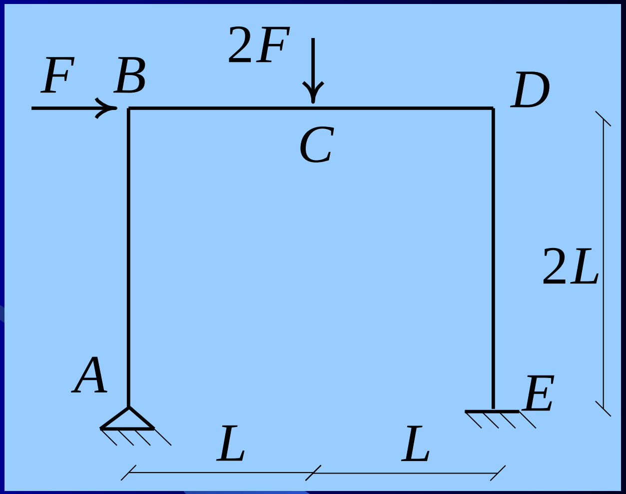
$$D_{\text{int}} = \dot{W}_{ext}$$

$$M_0(2\dot{\theta}) + M_0(2\dot{\theta}) + M_0(\dot{\theta}) = -F(\dot{\theta}2L) + 2F(\dot{\theta}L)$$

$$F = \infty$$



Limit Analysis, Kinematic Approach
Example, continue…:



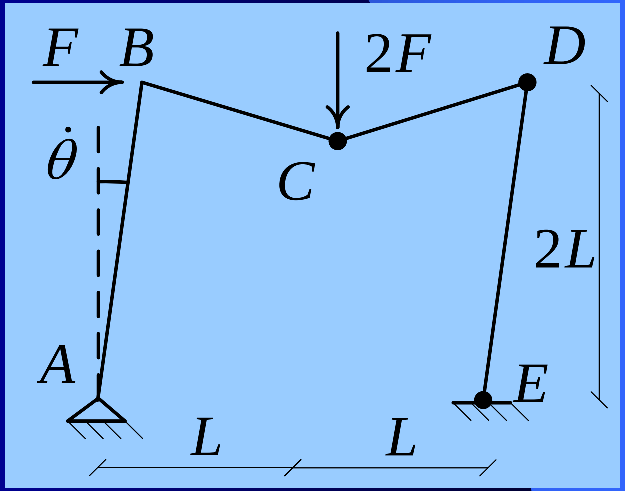
-Alternative 1: $F = \dfrac{2M_0}{L}$

-Alternative 2: $F = \dfrac{1.5M_0}{L}$

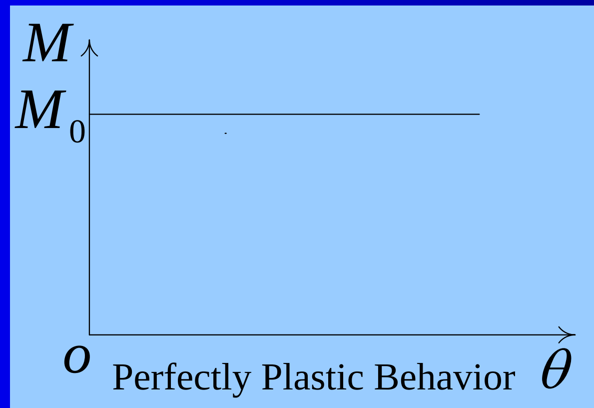
-Alternative 3: $F = \dfrac{1.25M_0}{L}$

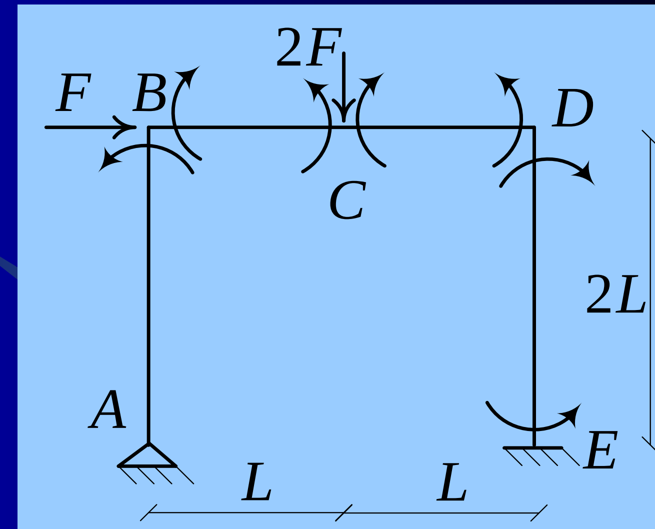
-Alternative 4: $F = \infty$

This is the true mechanism.



Limit Analysis in Frames, Static Approach, Example:



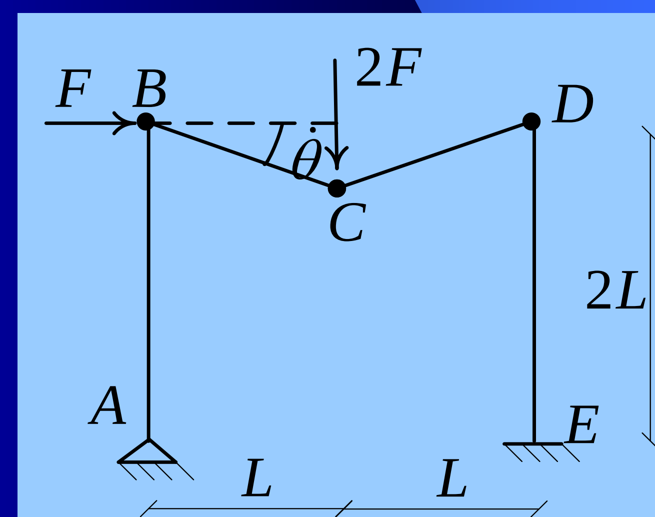


-The frame has two degree of freedom.

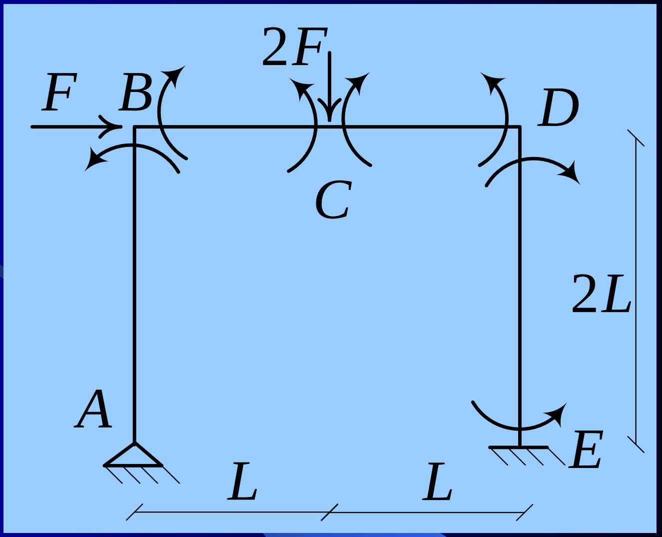
-Equilibrium equations using virtual work:

$$-M_B(\dot{\theta})+M_C(2\dot{\theta})-M_D(\dot{\theta})=2F(\dot{\theta}L)$$

$$M_B-2M_C+M_D=-2FL$$

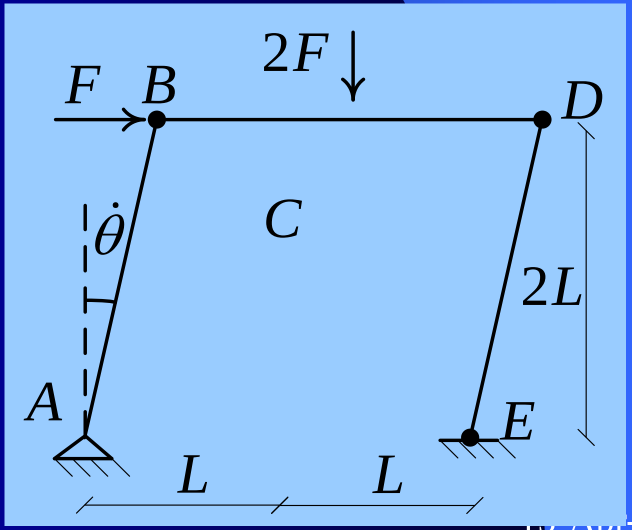


Limit Analysis in Frames,
Static Approach, Example, continue…:



$$M_B(\dot{\theta}) - M_D(\dot{\theta}) + M_E(\dot{\theta}) = F(\dot{\theta}2L)$$

$$-M_B + M_D - M_E = -2FL$$



Limit Analysis in Frames,
Static Approach, Example, continue…:

$$Equilibrium\ Equations: \begin{cases} M_B - 2M_C + M_D = -2FL \\ -M_B + M_D - M_E = -2FL \end{cases}$$

$$\begin{cases} M_C = 0.5M_B + 0.5M_D + FL \\ M_E = -M_B + M_D + 2FL \end{cases}$$

-The condition of plastic admissibility.

$$\begin{cases} -M_0 \le M_B \le M_0 \\ -M_0 \le M_C \le M_0 \\ -M_0 \le M_D \le M_0 \\ -M_0 \le M_E \le M_0 \end{cases}$$

$$\begin{cases} -M_0 \le M_B \le M_0 \\ -M_0 \le 0.5M_B + 0.5M_D + FL \le M_0 \\ -M_0 \le M_D \le M_0 \\ -M_0 \le -M_B + M_D + 2FL \le M_0 \end{cases}$$

Limit Analysis in Frames, Static Approach, Example, continue…:

$$\begin{cases} -M_0 \le M_B \le M_0 \\ -\dfrac{M_0}{L} - \dfrac{0.5M_B}{L} - \dfrac{0.5M_D}{L} \le F \le \dfrac{M_0}{L} - \dfrac{0.5M_B}{L} - \dfrac{0.5M_D}{L} \\ -M_0 \le M_D \le M_0 \\ -\dfrac{0.5M_0}{L} + \dfrac{0.5M_B}{L} - \dfrac{0.5M_D}{L} \le F \le \dfrac{0.5M_0}{L} + \dfrac{0.5M_B}{L} - \dfrac{0.5M_D}{L} \end{cases}$$

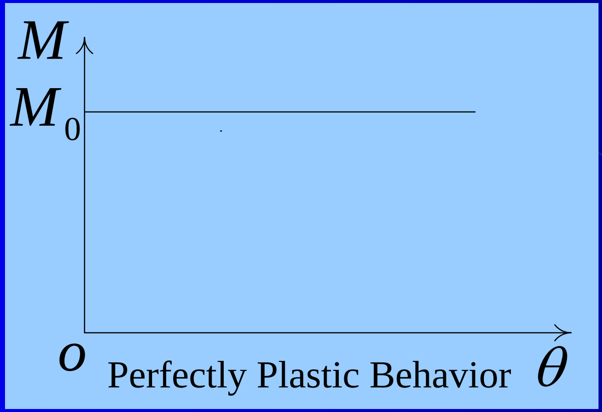
$$\begin{cases} -\dfrac{1.5M_0}{L} - \dfrac{M_D}{L} \le 2F \le \dfrac{1.5M_0}{L} - \dfrac{M_D}{L} \\ -M_0 \le M_D \le M_0 \end{cases}$$

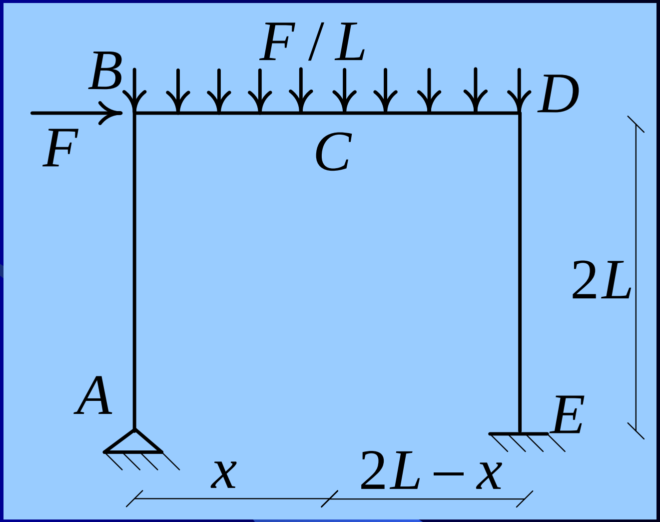
$$-\frac{2.5M_0}{L} \le 2F \le \frac{2.5M_0}{L}$$

$$-\frac{1.25M_0}{L} \le F \le \frac{1.25M_0}{L}$$

$$F_0 = \pm\frac{1.25M_0}{L}$$

Limit Analysis in Frames, Kinematic Approach, Example:





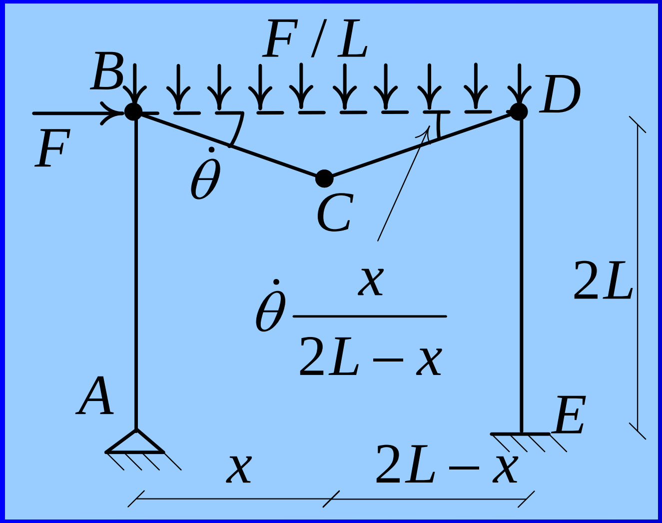
-The frame is statically indeterminate to the second degree and has four critical sections as B, C, D and E, so collapse occurred if the three hinges are inserted in these sections.

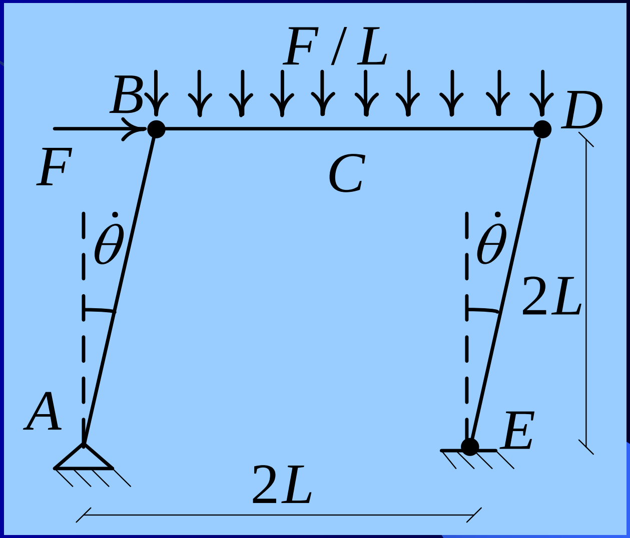
-Number of potential mechanisms:

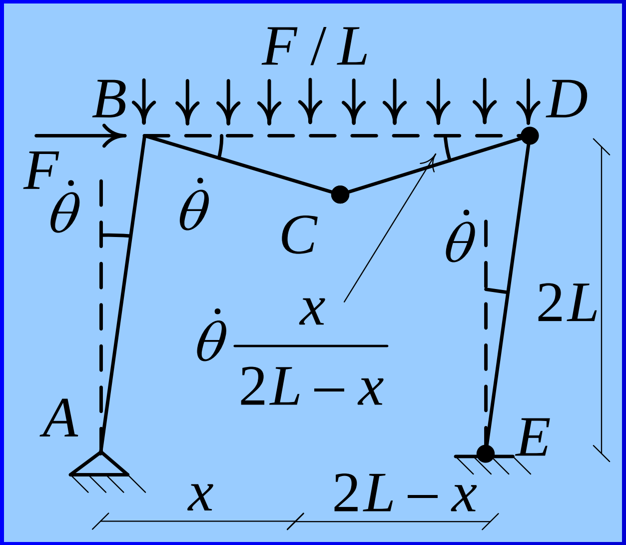
$$\binom{4}{3} = \frac{(4!)}{(3!)(1!)} = 4$$

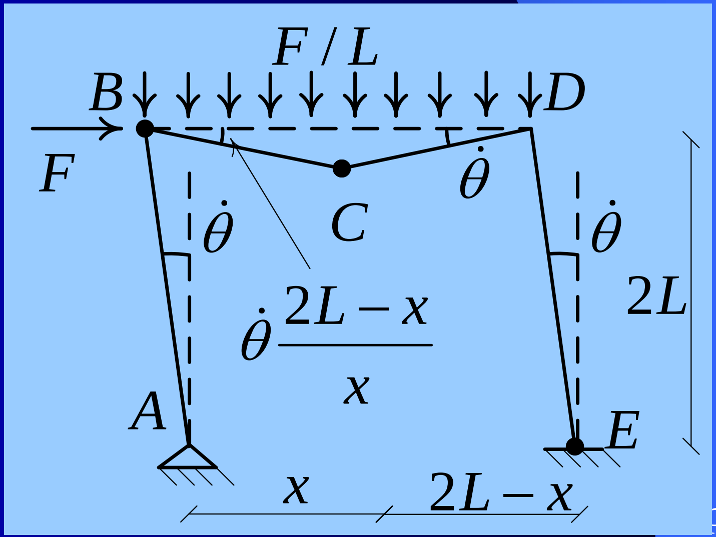
Limit Analysis in Frames, Kinematic Approach, Example, continue…:

-The potential mechanisms:

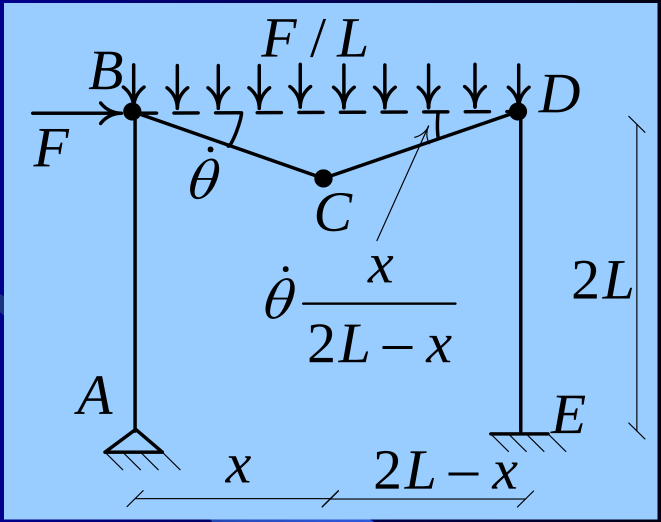








Limit Analysis in Frames, Kinematic Approach, Example, continue…:



-Alternative 1:

$$D_{\text{int}} = \dot{W}_{ext}$$

$$M_0(\dot{\theta}) + M_0\left(\dot{\theta} + \dot{\theta}\frac{x}{2L-x}\right) + M_0\left(\dot{\theta}\frac{x}{2L-x}\right) = (F/L)\frac{(\dot{\theta}x)(2L)}{2}$$

$$F = 2M_0\left(\frac{1}{x} + \frac{1}{2L-x}\right)$$

$$F = \frac{4M_0 L}{x(2L-x)}$$

$$\frac{dF}{dx} = 0$$

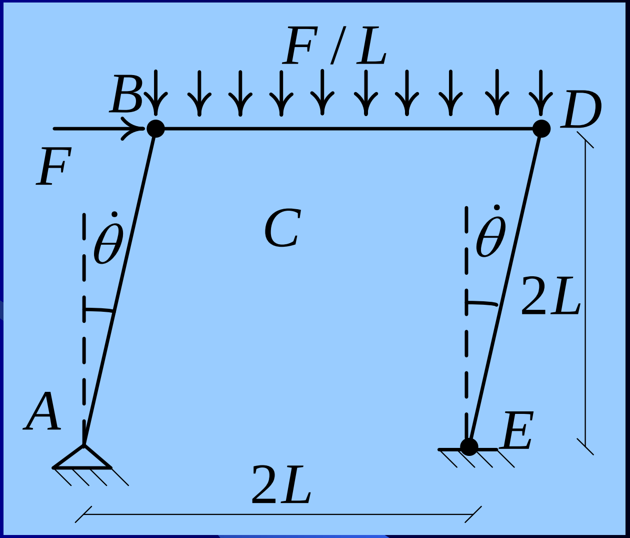
$$\frac{-(2L-2x)}{x^2(2L-x)^2} = 0$$

$$x = L$$

$$F_k = \frac{4M_0}{L}$$

Limit Analysis in Frames,
Kinematic Approach, Example, continue…:

-Alternative 2:



$$D_{\text{int}} = \dot{W}_{ext}$$

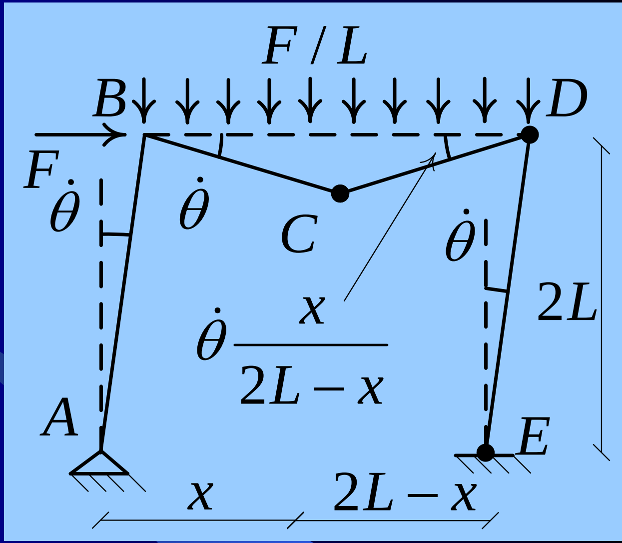
$$M_0(\dot{\theta}) + M_0(\dot{\theta}) + M_0(\dot{\theta}) = F\dot{\theta}(2L)$$

$$F_k = \frac{1.5M_0}{L}$$

Limit Analysis in Frames,
Kinematic Approach, Example, continue…:

-Alternative 3:

$$D_{\text{int}} = \dot{W}_{ext}$$



$$M_0\left(\dot\theta + \dot\theta\frac{x}{2L-x}\right) + M_0\left(\dot\theta + \dot\theta\frac{x}{2L-x}\right) + M_0(\dot\theta) = F\dot\theta(2L) + (F/L)\frac{(\dot\theta x)(2L)}{2}$$

$$F = \frac{M_0}{2L+x}\left(3 + \frac{2x}{2L-x}\right)$$

$$F = \frac{M_0(6L-x)}{4L^2 - x^2}$$

$$\frac{dF}{dx} = 0$$

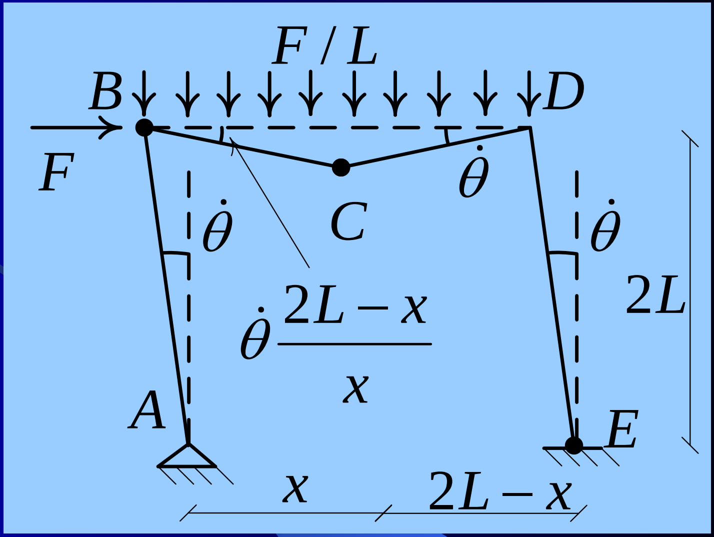
$$\frac{-4L^2 + x^2 + 12Lx - 2x^2}{(4L^2 - x^2)^2} = 0$$

$$x = 0.343L$$

$$F_k = \frac{1.457M_0}{L}$$

Limit Analysis in Frames, Kinematic Approach, Example, continue…:

-Alternative 4:

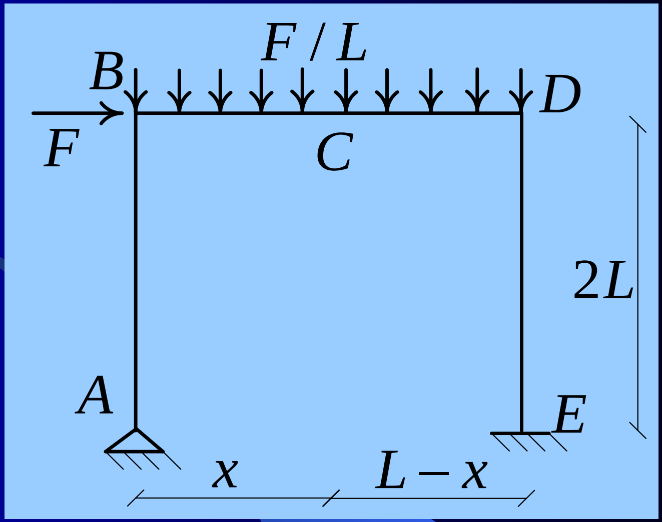


$$D_{\text{int}} = \dot{W}_{ext}$$

$$M_0(\dot{\theta}+\dot{\theta}\frac{2L-x}{x})+M_0(\dot{\theta}+\dot{\theta}\frac{2L-x}{x})+M_0(\dot{\theta})=-F\dot{\theta}(2L)+(F/L)\frac{\dot{\theta}(2L-x)(2L)}{2}$$

$$\dot{W}_{ext}=-F\dot{\theta}x<0$$ → This isn't an admissible mechanism.

Limit Analysis, Kinematic Approach
Example, continue…:



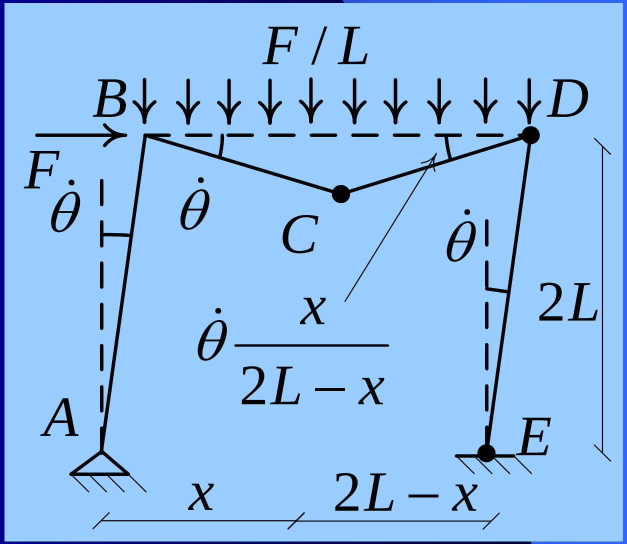
-Alternative 1: $F = \dfrac{4M_0}{L}$

-Alternative 2: $F = \dfrac{1.5M_0}{L}$

-Alternative 3: $F = \dfrac{1.457M_0}{L}$

This is the true mechanism.

-Alternative 4 is impossible.



Elementary and Combined Mechanisms:

-Some of the mechanisms, among all the potential mechanisms, are elementary and the others are combined.

- The elementary mechanisms have two following properties:

1) No elementary mechanism can be obtained as a linear combination of the others.

2) Any potential mechanism can be obtained as a linear combination of the elementary ones.

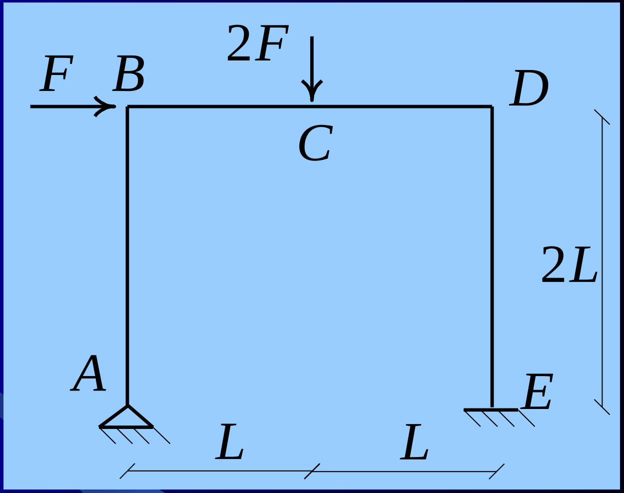
- The number of elementary mechanisms : n

- The number of critical sections or the number of bars : m

- The degree of static redundancy : s

$$n = m - s$$

Elementary and Combined Mechanisms, Example:



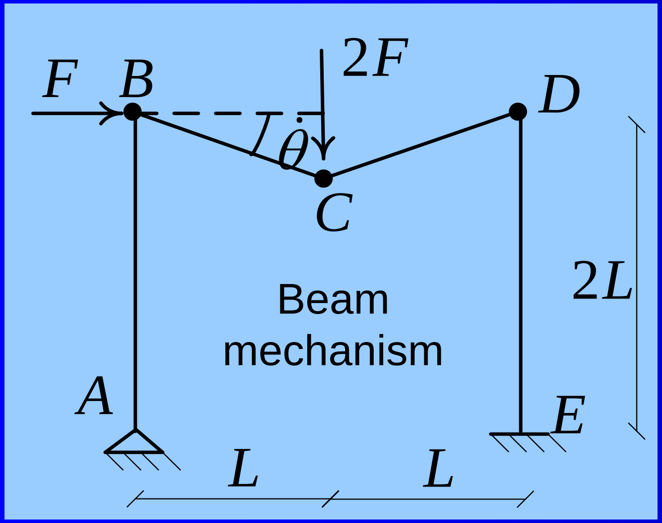
- The number of critical sections: m=4

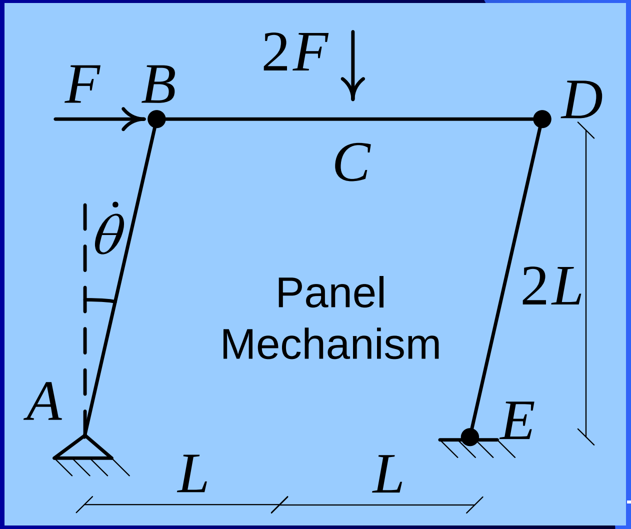
- The degree of static redundancy: s=2

- The number of elementary mechanisms : n=4-2=2

- The number of all potential mechanisms: $\binom{4}{3} = \dfrac{(4!)}{(3!)(1!)} = 4$

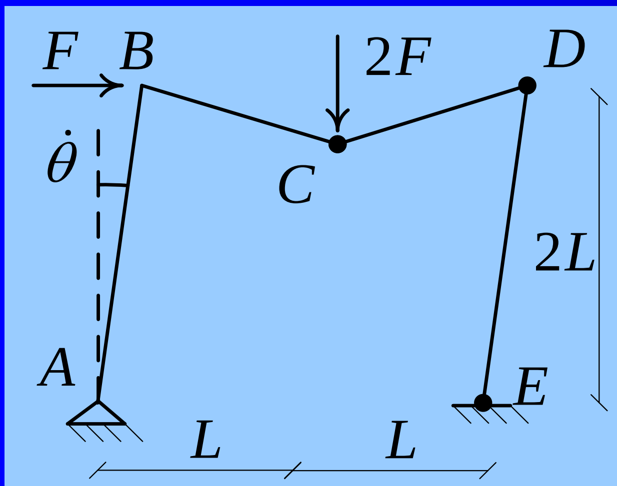
- The elementary mechanisms:





Elementary and Combined Mechanisms, Example, continue…:

- The combined mechanisms:



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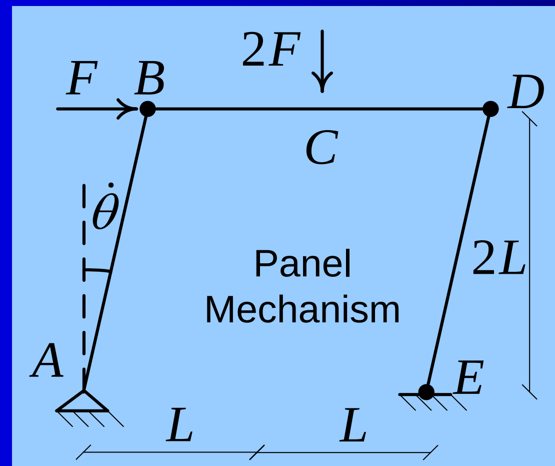

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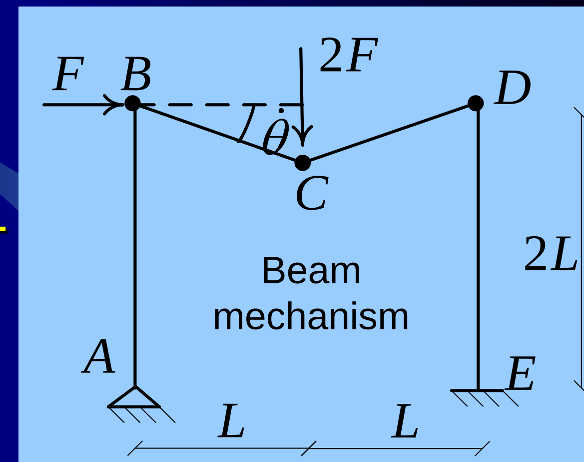

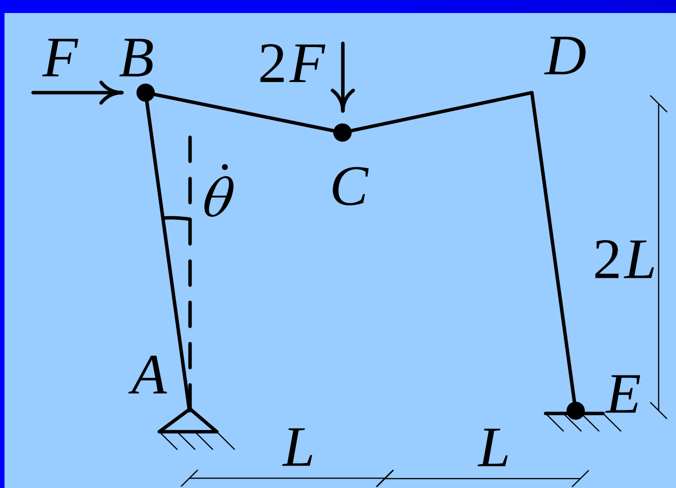

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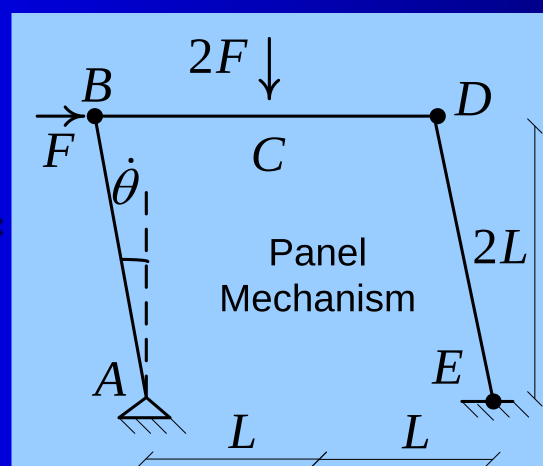

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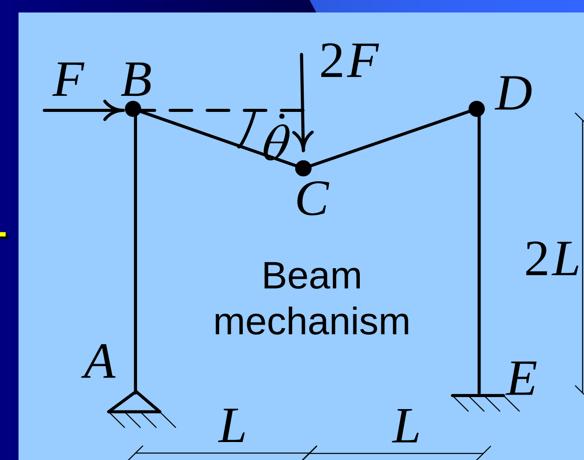
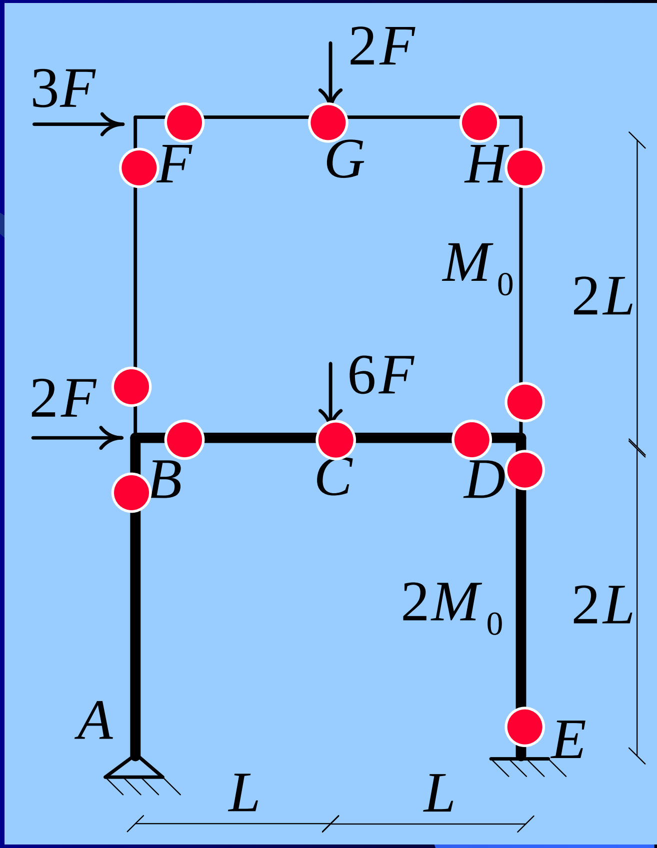


Limit Analysis in Frames,
, Kinematic Approach, Example:



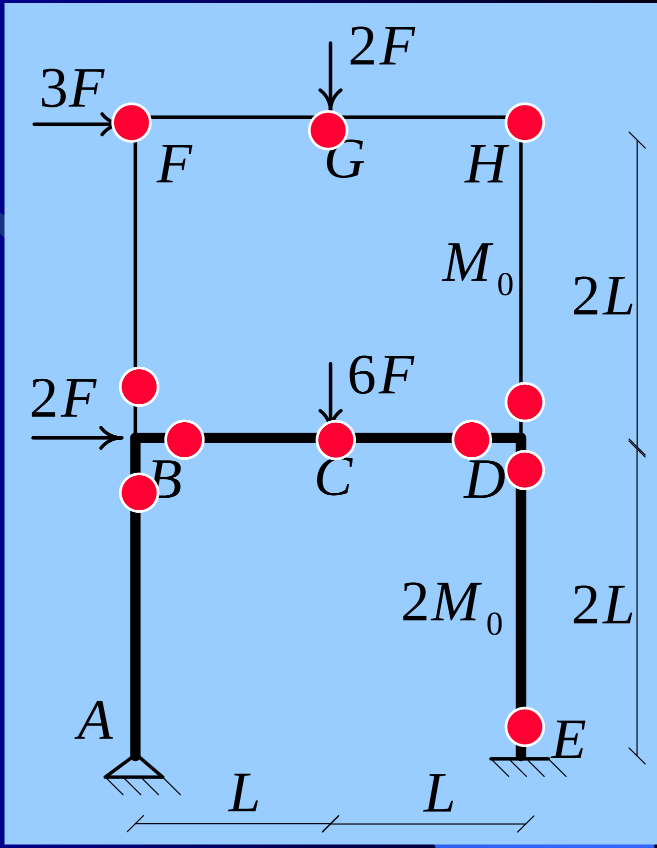
First Approach:

- The number of critical sections: m=13

- The degree of static redundancy: s=5

- The number of elementary mechanisms : n=13-5=8

Limit Analysis in Frames,
, Kinematic Approach, Example, continue…:



Second Approach:

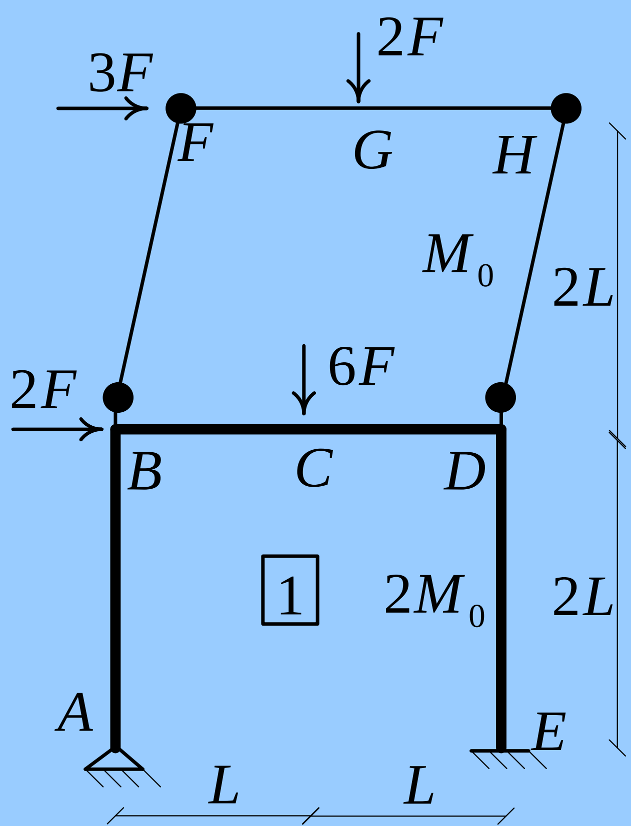
- The number of critical sections: m=11

- The degree of static redundancy: s=5

- The number of elementary mechanisms : n=11-5=6

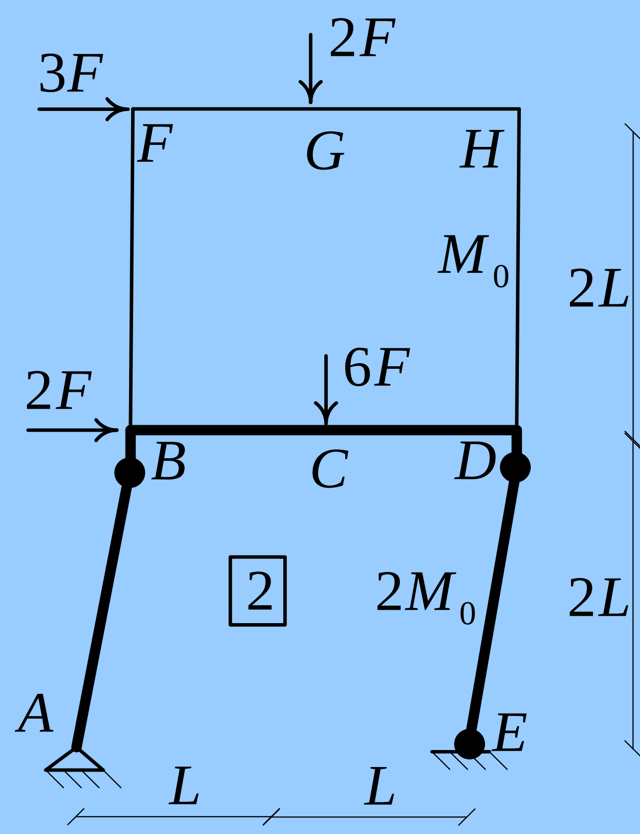
Limit Analysis in Frames,
, Kinematic Approach, Example, continue…:

The six elementary mechanisms:



Panel mechanism

$$F_k = \frac{2M_0}{3L}$$

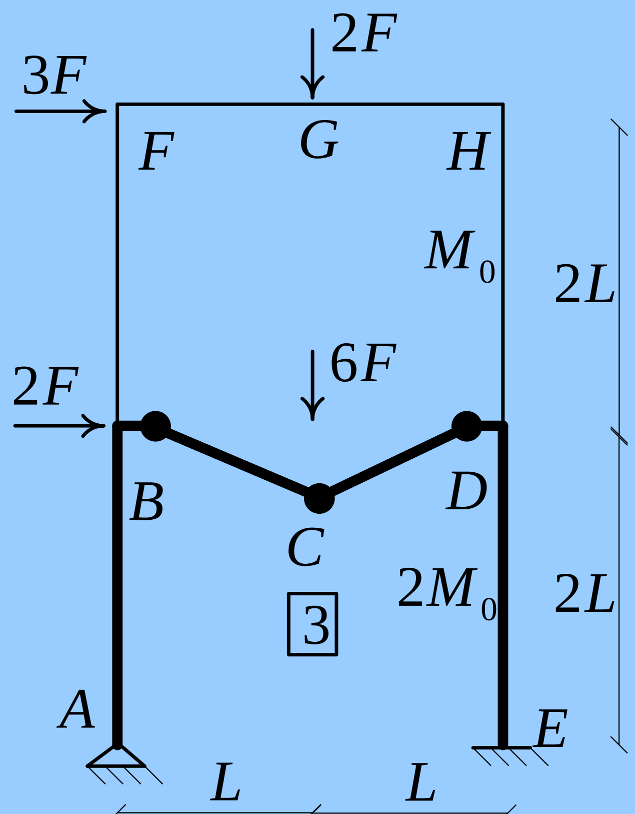


Panel mechanism

$$F_k = \frac{3M_0}{5L}$$

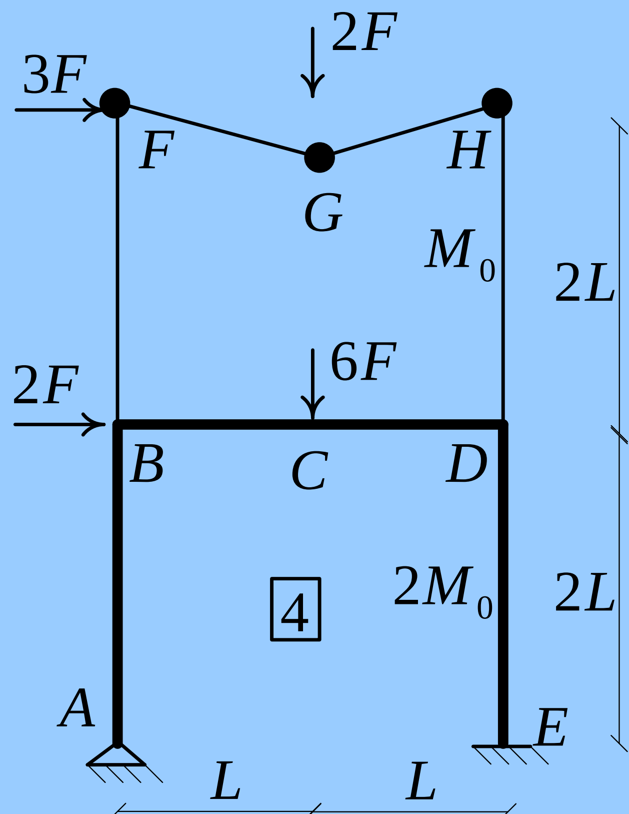
Limit Analysis in Frames, , Kinematic Approach, Example, continue…:

The six elementary mechanisms, continue…:



Beam mechanism

$$F_k = \frac{4M_0}{3L}$$

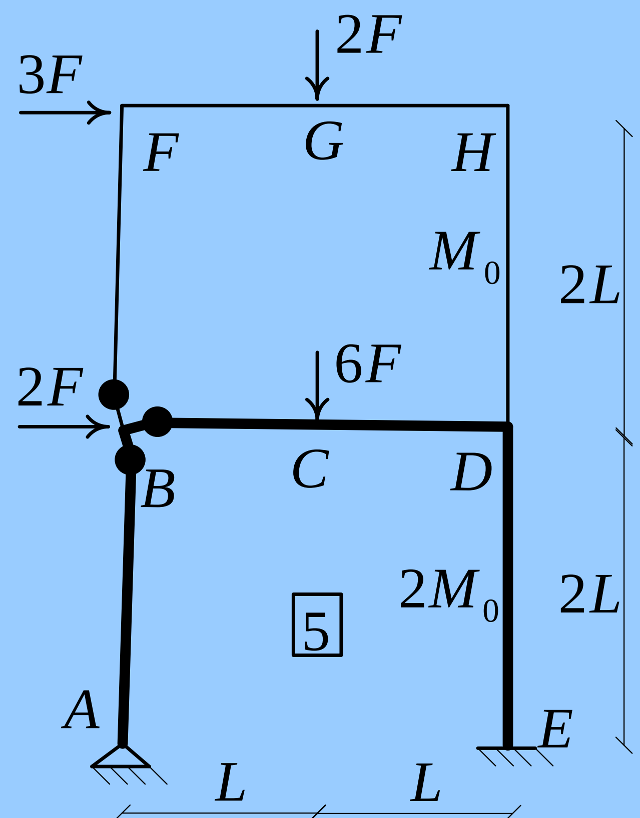


Beam mechanism

$$F_k = \frac{2M_0}{L}$$

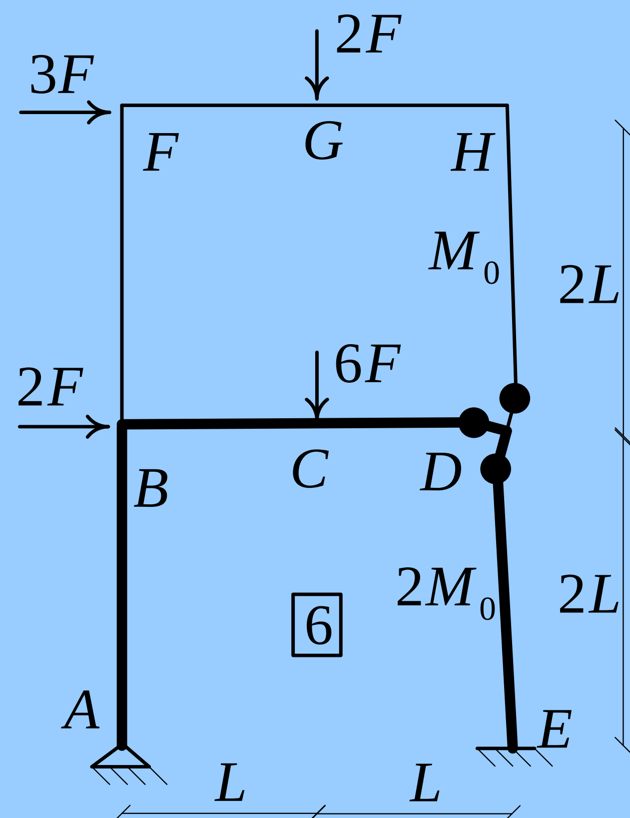
Limit Analysis in Frames, , Kinematic Approach, Example, continue…:

The six elementary mechanisms, continue…:



Joint mechanism

$F_k = \infty$

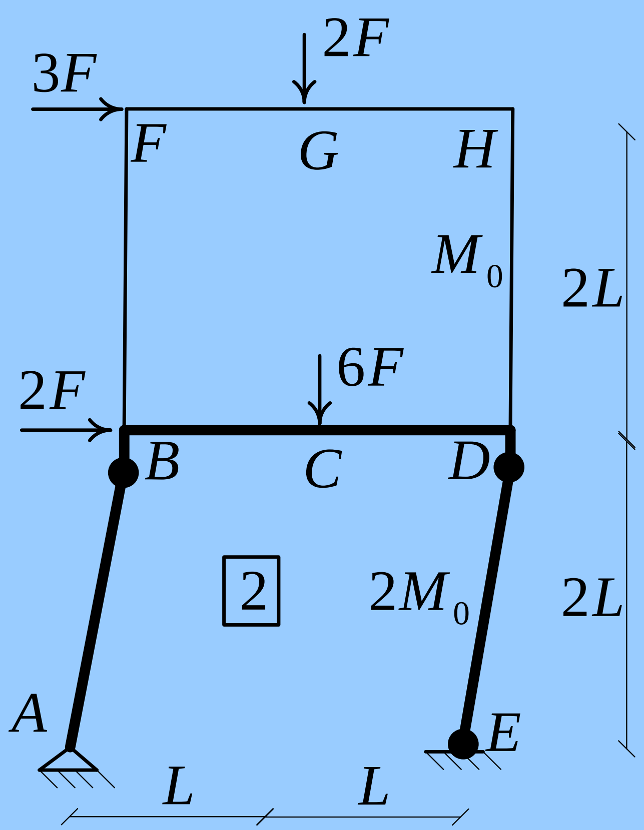


Joint mechanism

$F_k = \infty$

Limit Analysis in Frames, , Kinematic Approach, Example, continue…:

The most critical mechanism among elementary mechanisms:

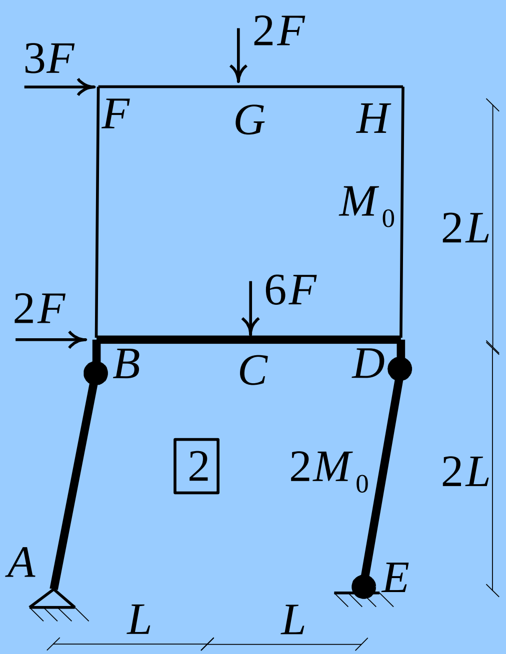


Panel mechanism

$$F_k = \frac{3M_0}{5L}$$

Limit Analysis in Frames, , Kinematic Approach, Example, continue…:

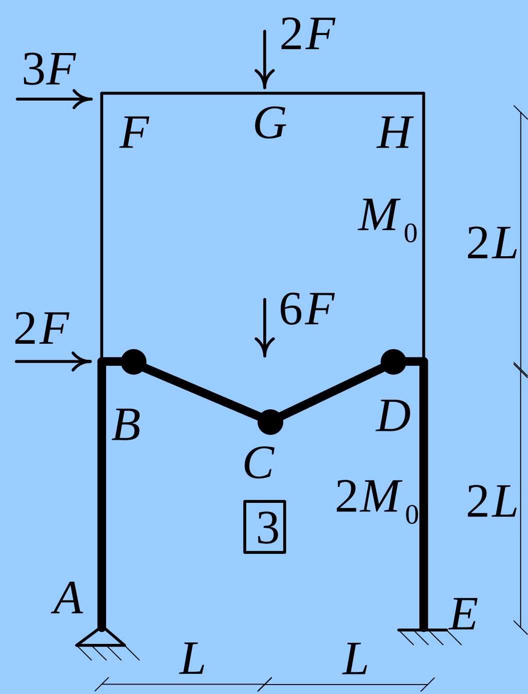
The combined mechanisms:



Panel mechanism

$$F_k = \frac{3M_0}{5L}$$

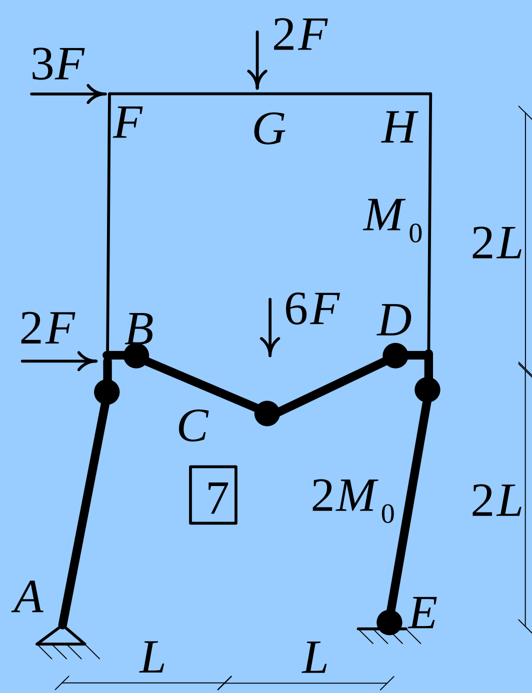
+



Beam mechanism

$$F_k = \frac{4M_0}{3L}$$

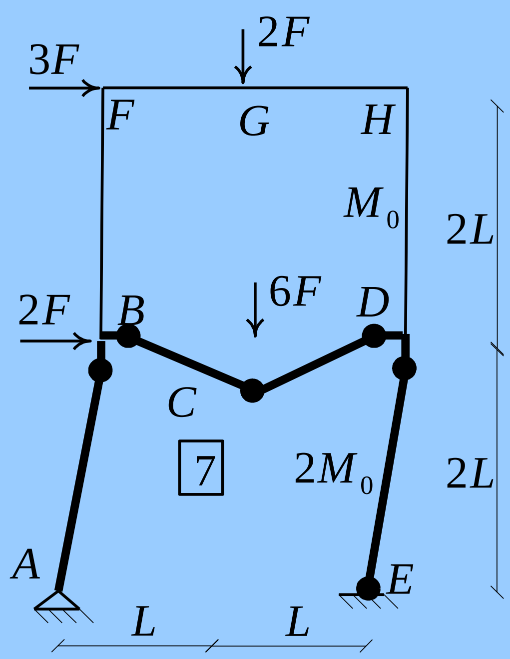
=



$$F_k = \frac{7M_0}{8L}$$

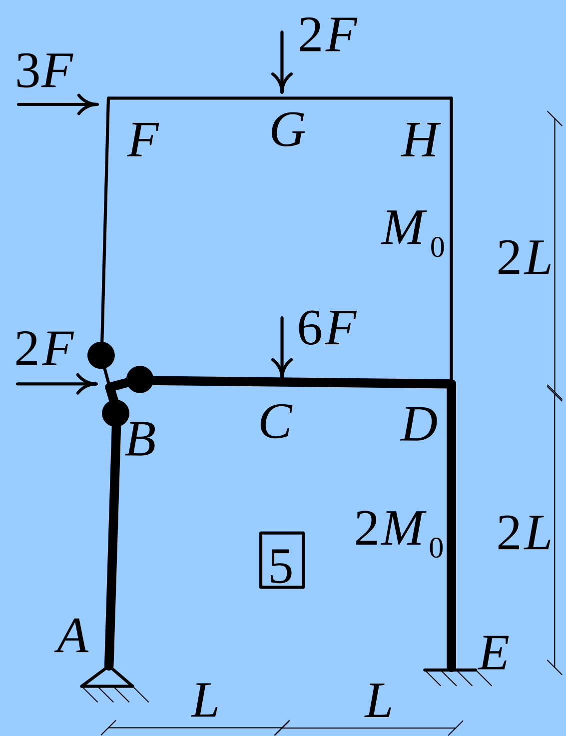
Limit Analysis in Frames,
, Kinematic Approach, Example, continue…:

The combined mechanisms:



$$F_k = \frac{7M_0}{8L}$$

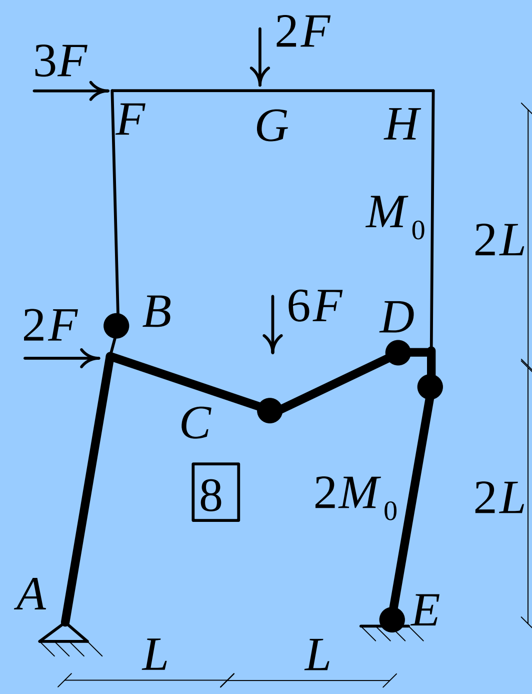
\-



Joint mechanism

$$F_k = \infty$$

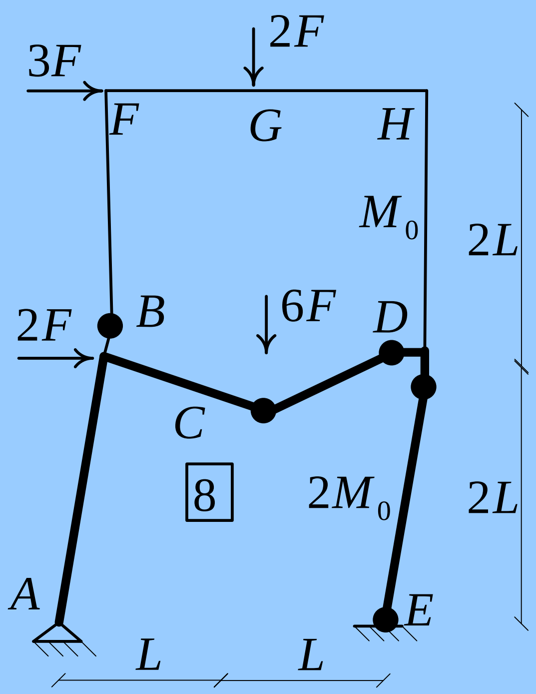
=



$$F_k = \frac{11M_0}{16L}$$

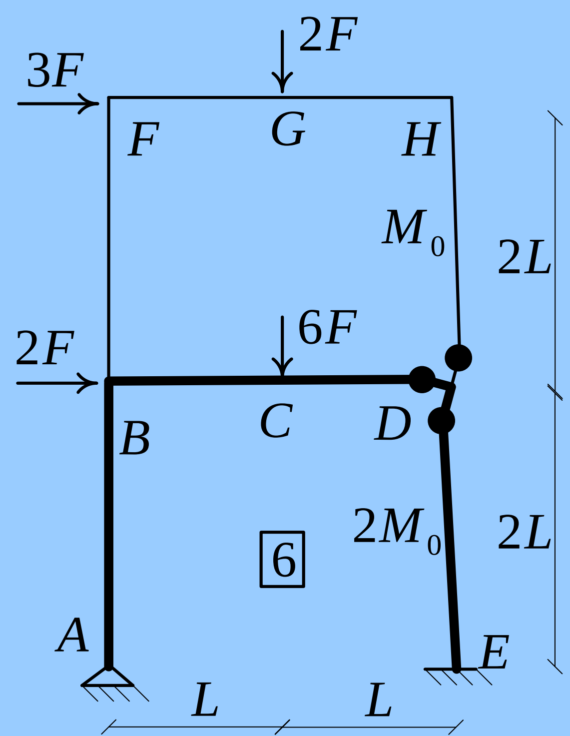
Limit Analysis in Frames, , Kinematic Approach, Example, continue…:

The combined mechanisms:



$$F_k = \frac{11M_0}{16L}$$

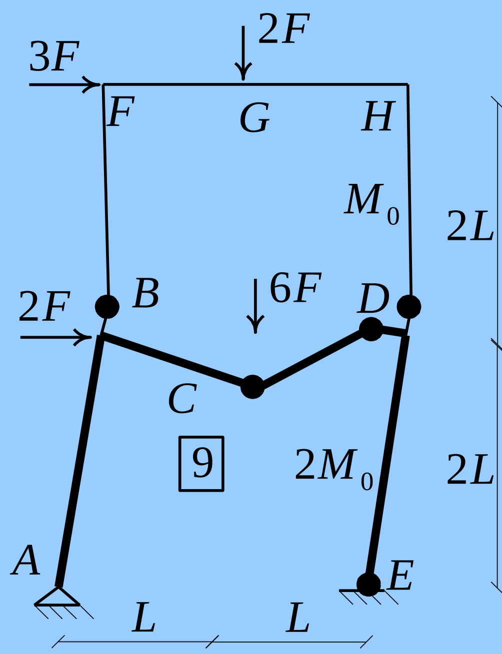
+



Joint mechanism

$$F_k = \infty$$

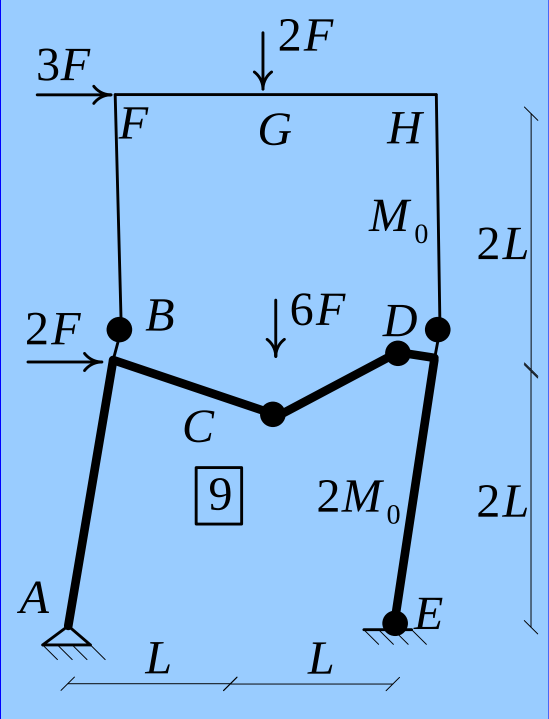
=



$$F_k = \frac{12M_0}{16L}$$

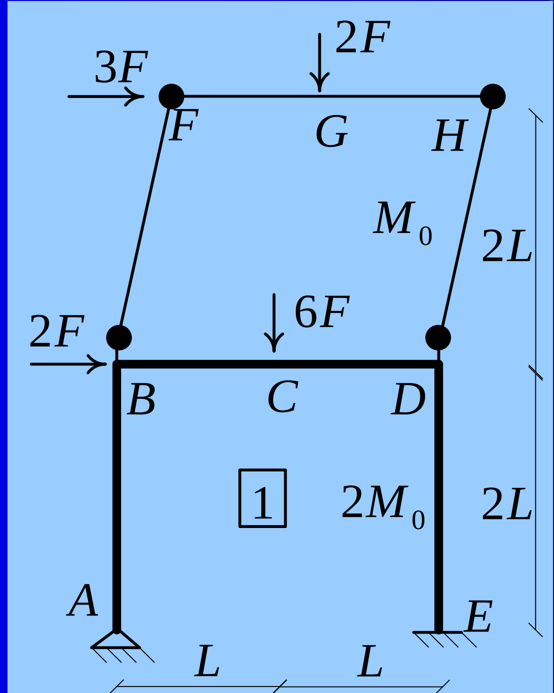
Limit Analysis in Frames, , Kinematic Approach, Example, continue…:

The combined mechanisms:



$$F_k = \frac{12M_0}{16L}$$

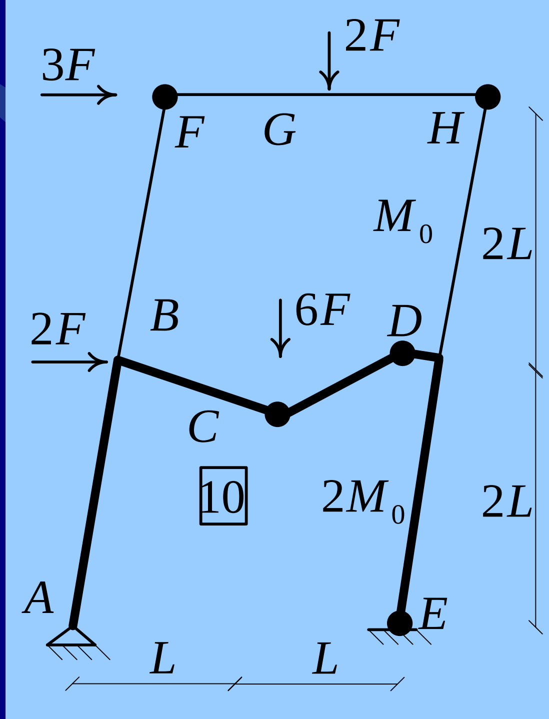
+



Panel mechanism

$$F_k = \frac{2M_0}{3L}$$

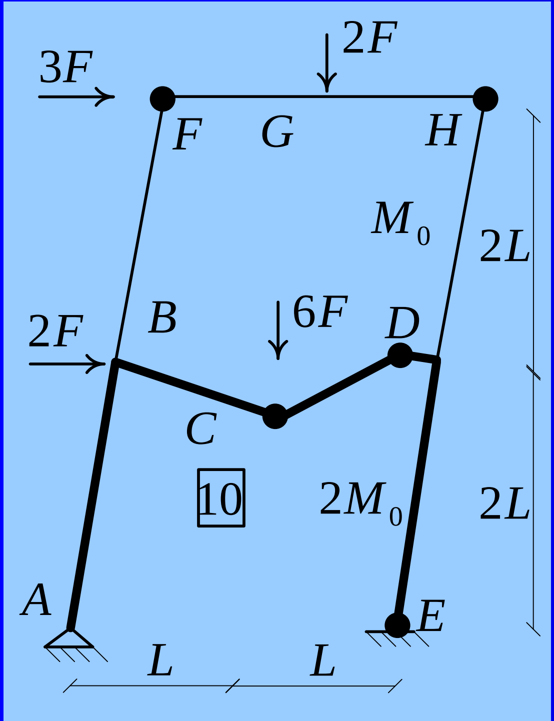
=



$$F_k = \frac{6M_0}{11L}$$

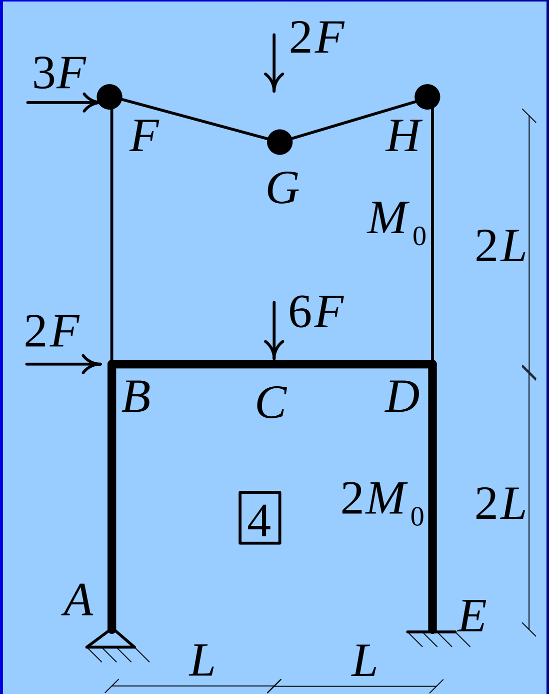
Limit Analysis in Frames,
, Kinematic Approach, Example, continue…:

The combined mechanisms:



$$F_k = \frac{6M_0}{11L}$$

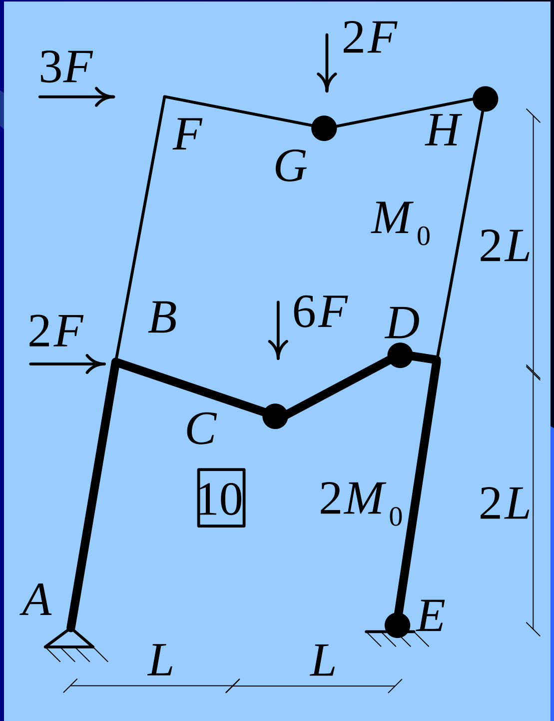
+



Beam mechanism

$$F_k = \frac{2M_0}{L}$$

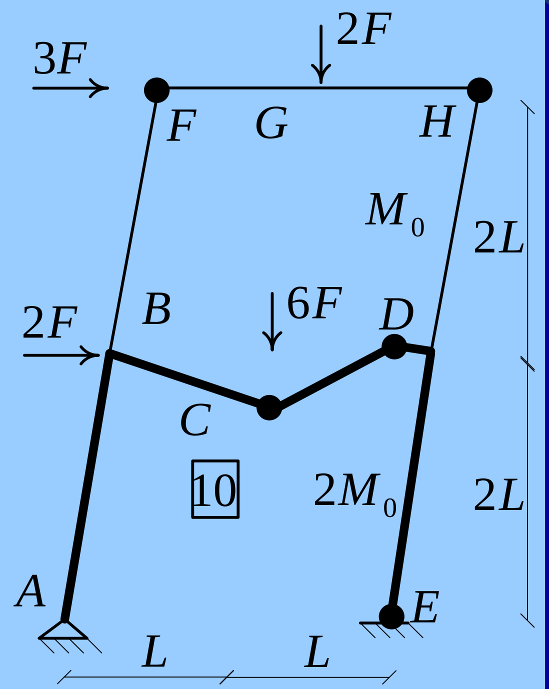
=



$$F_k = \frac{7M_0}{12L}$$

Limit Analysis in Frames,
, Kinematic Approach, Example, continue…:

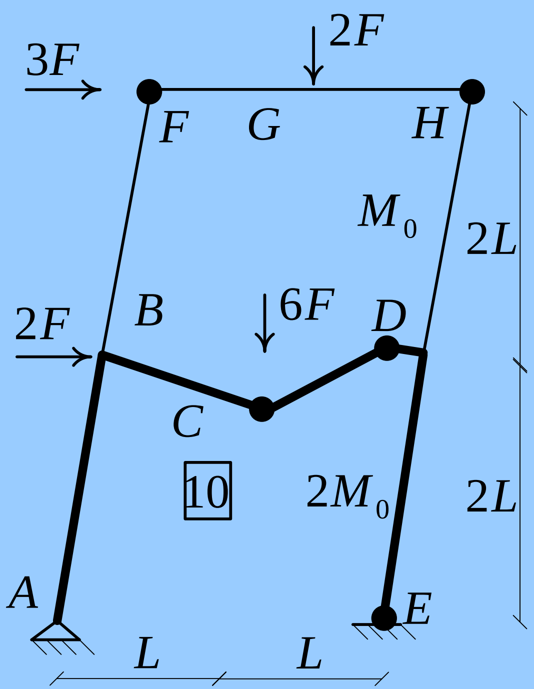
The most critical mechanism among all of the examined mechanisms:



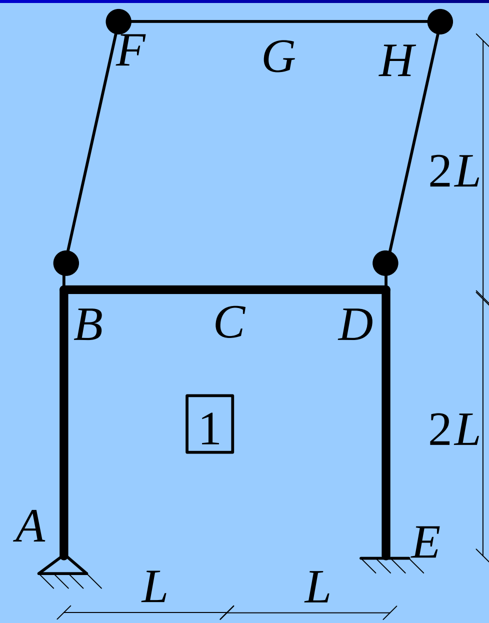
$$F_k = \frac{6M_0}{11L}$$

Limit Analysis in Frames,
, Kinematic Approach, Example, continue…:

-Equilibrium equations using virtual work:



$$F_k = \frac{6M_0}{11L}$$

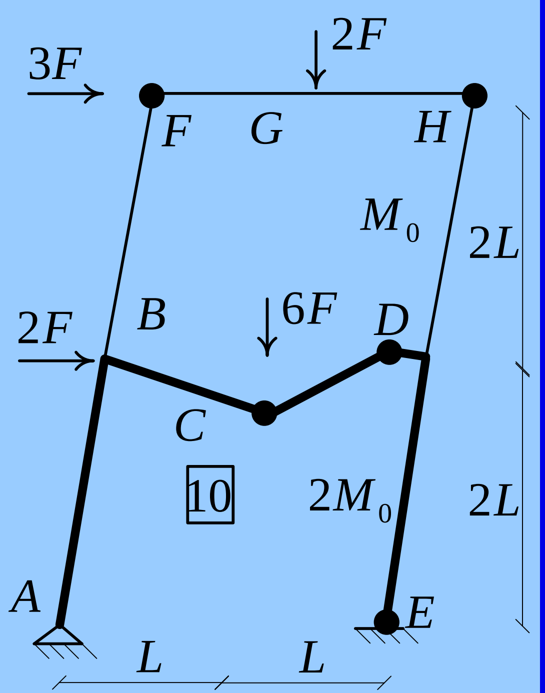


$$M_{BT}(-1) + M_0(1) + M_0(1) + M_{DT}(+1) = 3F(2L)$$

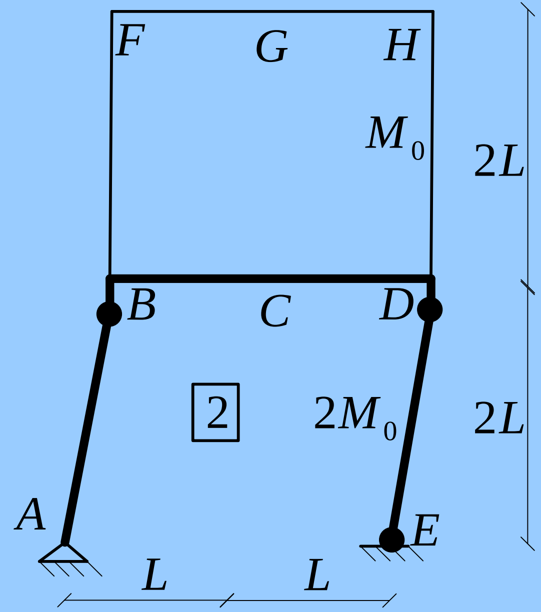
$$-M_{BT} + M_{DT} = \frac{14M_0}{11}$$

Limit Analysis in Frames,
, Kinematic Approach, Example, continue…:

-Equilibrium equations using virtual work, continue…:



$$F_k = \frac{6M_0}{11L}$$



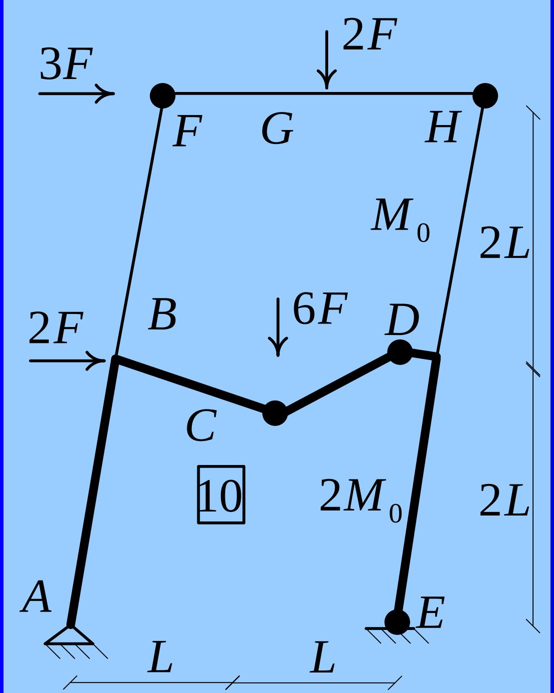
Panel mechanism

$$M_{BB}(+1) + M_{DB}(-1) + M_0(1) = 3F(2L) + 2F(2L)$$

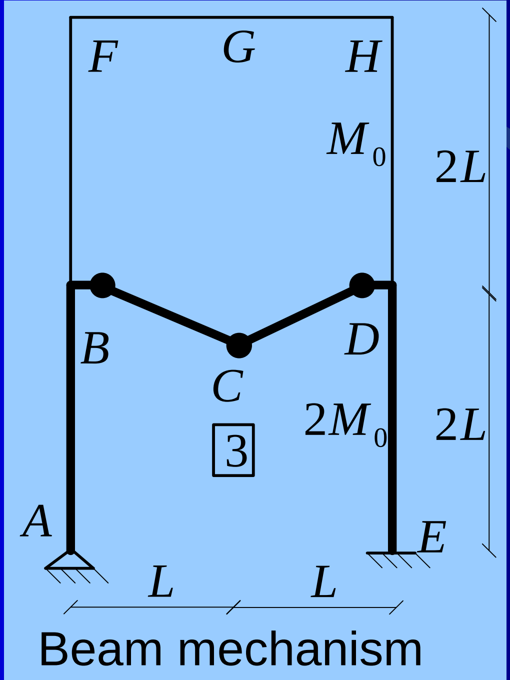
$$+M_{BB} - M_{DB} = \frac{49M_0}{11}$$

Limit Analysis in Frames,
, Kinematic Approach, Example, continue…:

-Equilibrium equations using virtual work, continue…:



$$F_k = \frac{6M_0}{11L}$$



$$M_{BR}(-1) + 2M_0(2) + 2M_0(1) = 6F(L)$$

$$M_{BR} = +\frac{30}{11}M_0 > 2M_0$$

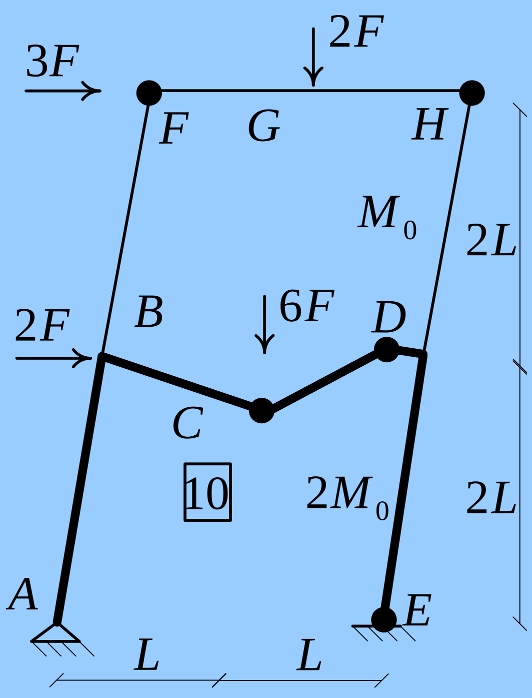
This mechanism is impossible.

$$\frac{6M_0}{11L}\,\frac{2}{30/11} < F_0 < \frac{6M_0}{11L}$$

$$\frac{2M_0}{5L} < F_0 < \frac{6M_0}{11L}$$

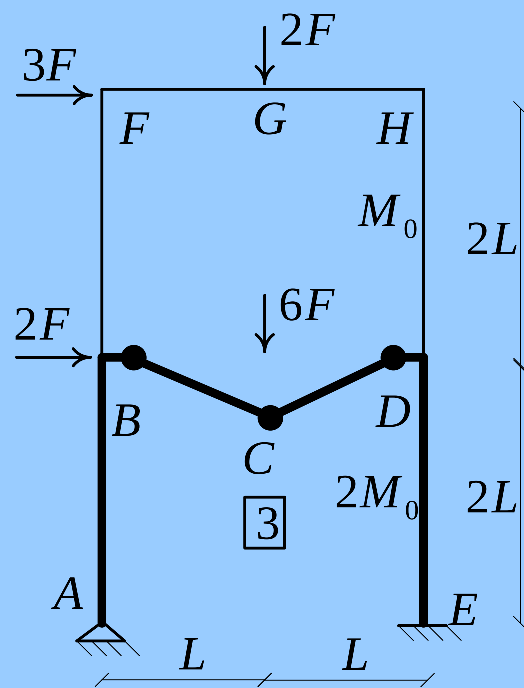
Limit Analysis in Frames, , Kinematic Approach, Example, continue…:

The combined mechanisms:



$$F_k = \frac{6M_0}{11L}$$

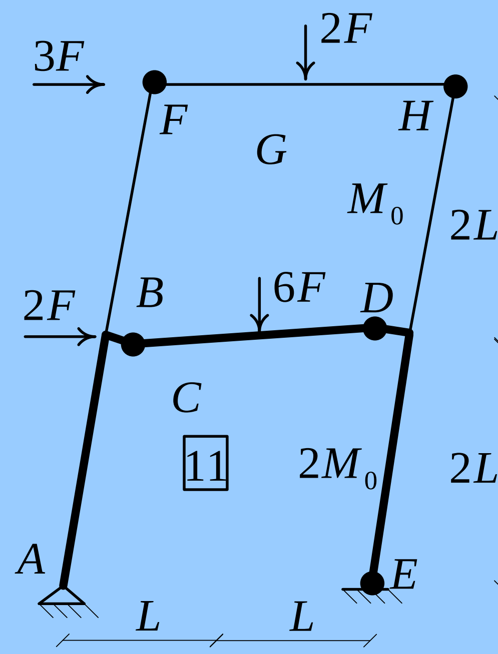
-



Beam mechanism

$$F_k = \frac{4M_0}{3L}$$

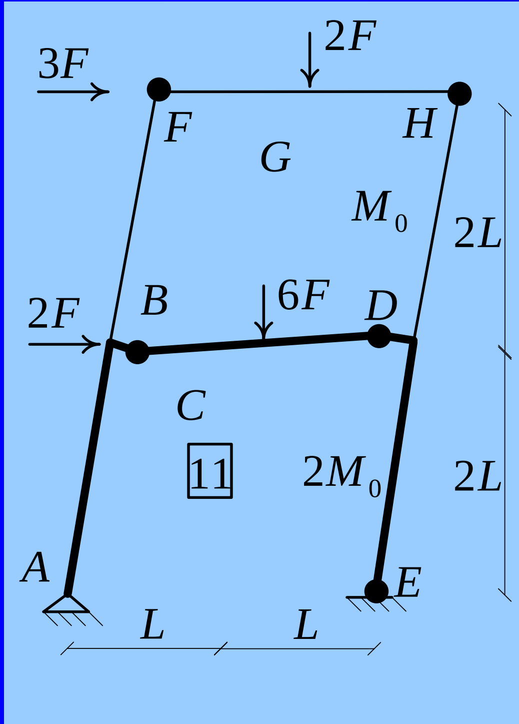
=



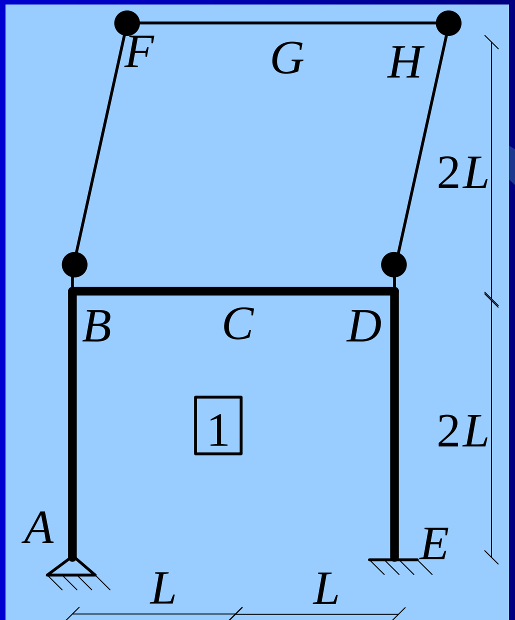
$$F_k = \frac{M_0}{2L}$$

Limit Analysis in Frames,
, Kinematic Approach, Example, continue…:

-Equilibrium equations using virtual work:



$$F_k = \frac{M_0}{2L}$$



Panel mechanism

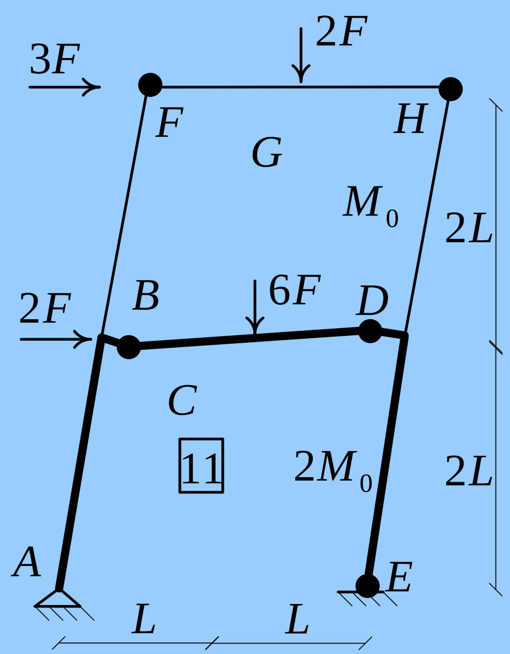
$$M_{BT}(-1) + M_0(1) + M_0(1) + M_{DT}(+1) = 3F(2L)$$

$$-M_{BT} + M_{DT} = M_0$$

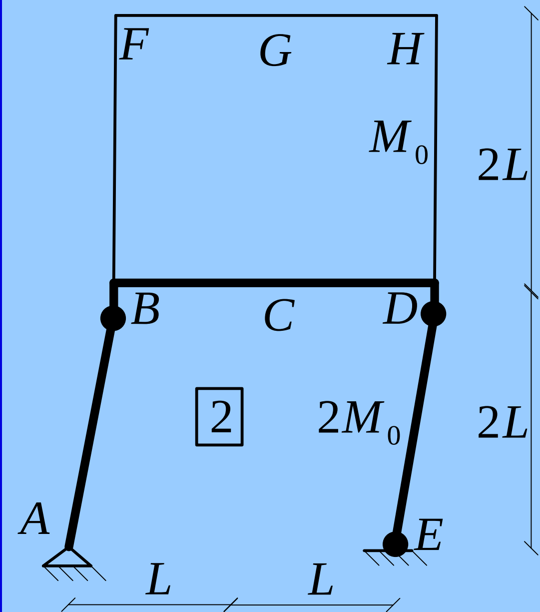
Limit Analysis in Frames,

, Kinematic Approach, Example, continue…:

-Equilibrium equations using virtual work, continue…:



$$F_k = \frac{M_0}{2L}$$



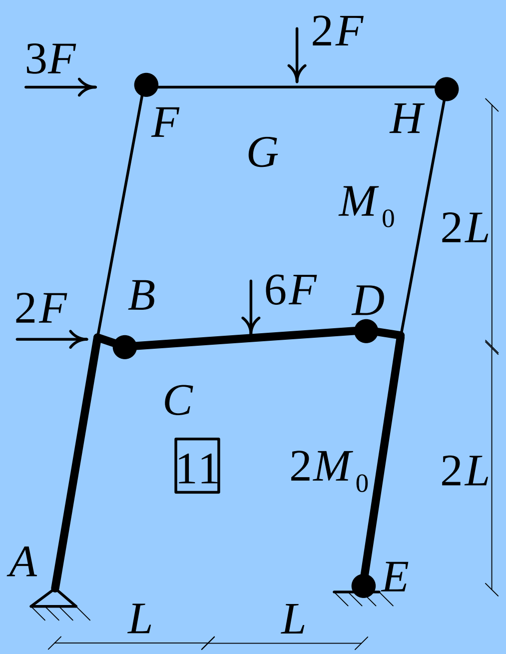
Panel mechanism

$$M_{BB}(+1) + M_{DB}(-1) + 2M_0(1) = 3F(2L) + 2F(2L)$$

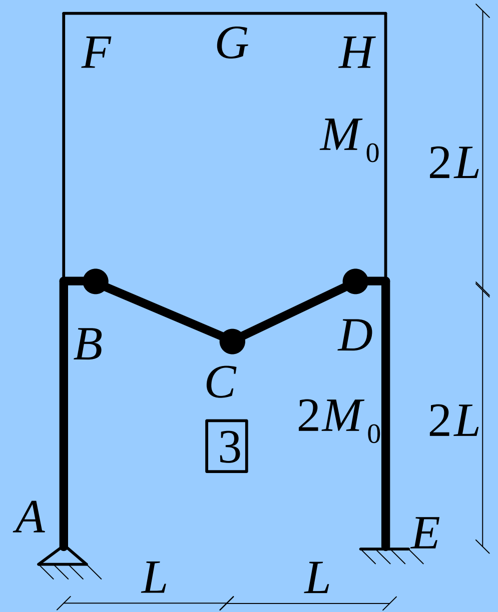
$$+M_{BB} - M_{DB} = 3M_0$$

Limit Analysis in Frames,
, Kinematic Approach, Example, continue…:

-Equilibrium equations using virtual work, continue…:



$$F_k = \frac{M_0}{2L}$$



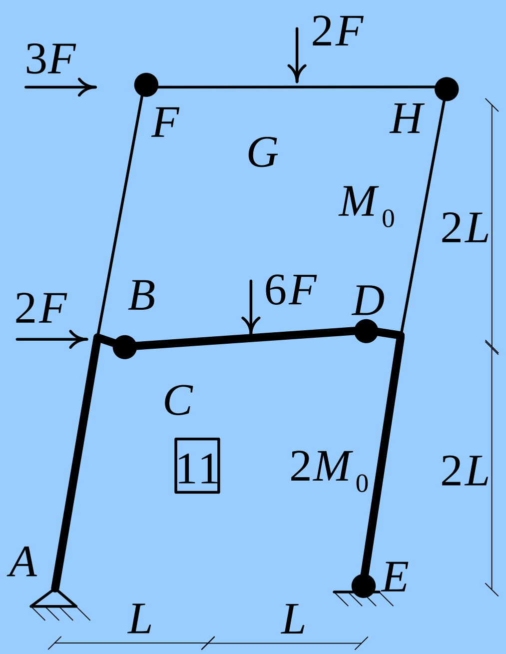
Beam mechanism

$$2M_0(-1) + M_C(2) + 2M_0(+1) = 6F(L)$$

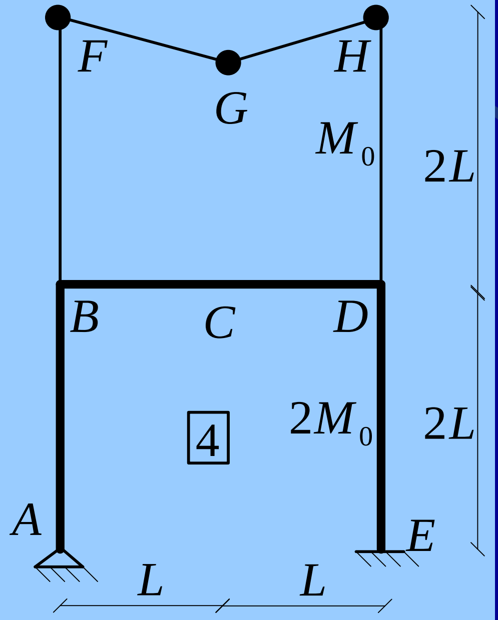
$$M_C = 1.5M_0$$

Limit Analysis in Frames,
, Kinematic Approach, Example, continue…:

-Equilibrium equations using virtual work, continue…:



$$F_k = \frac{M_0}{2L}$$



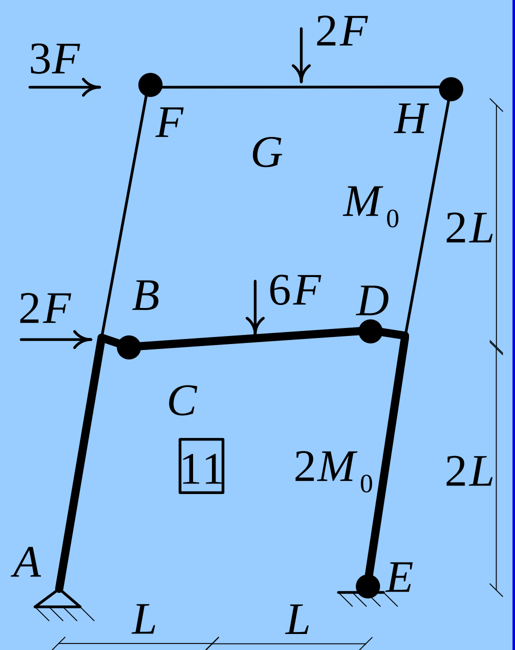
Beam mechanism

$$M_0(-1) + M_G(2) + M_0(+1) = 2F(L)$$

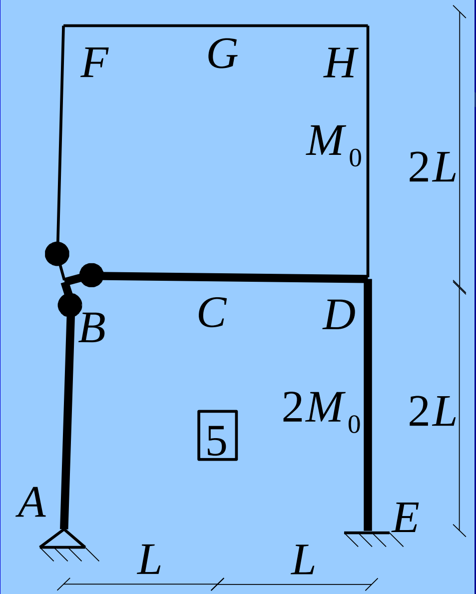
$$M_G = 0.5M_0$$

Limit Analysis in Frames, , Kinematic Approach, Example, continue…:

-Equilibrium equations using virtual work:



$$F_k = \frac{M_0}{2L}$$

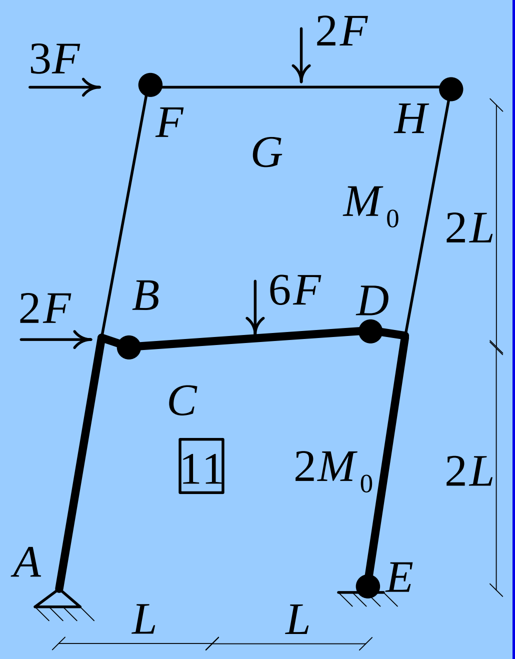


$$M_{BT}(-1) + 2M_0(-1) + M_{BB}(+1) = 0$$

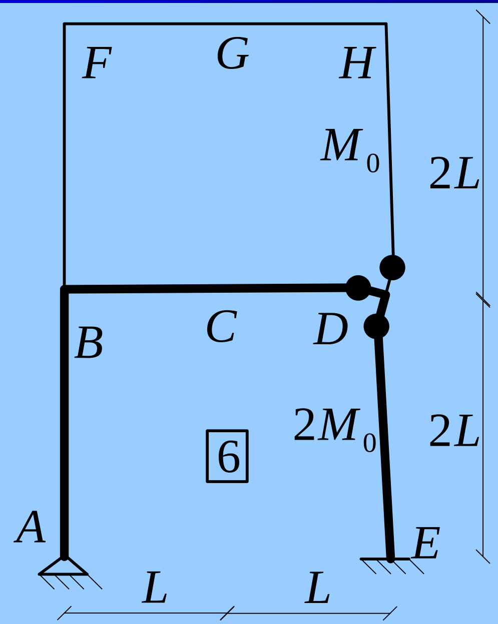
$$-M_{BT} + M_{BB} = 2M_0$$

Limit Analysis in Frames,
, Kinematic Approach, Example, continue…:

-Equilibrium equations using virtual work:



$$F_k = \frac{M_0}{2L}$$



Joint mechanism

$$M_{DT}(-1) + 2M_0(1) + M_{DB}(+1) = 0$$

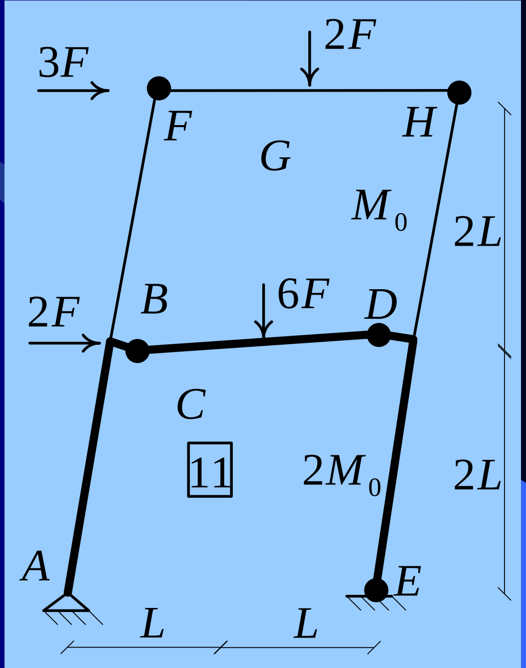
$$-M_{DT} + M_{DB} = -2M_0$$

Limit Analysis in Frames,
, Kinematic Approach, Example, continue…:

-Equilibrium equations:

$$\begin{cases} -M_{BT} + M_{DT} = M_0 \\ +M_{BB} - M_{DB} = 3M_0 \\ M_C = 1.5M_0 \\ M_G = 0.5M_0 \\ -M_{BT} + M_{BB} = 2M_0 \\ -M_{DT} + M_{DB} = -2M_0 \end{cases}$$

$$\begin{cases} M_{DT} = M_0 \\ M_{BT} = 0 \rightarrow suppose \\ M_C = 1.5M_0 \\ M_C = 0.5M_0 \\ M_{BB} = 2M_0 \\ M_{DB} = -M_0 \end{cases}$$



$$F_k = \frac{M_0}{2L}$$

This is the True mechanism:

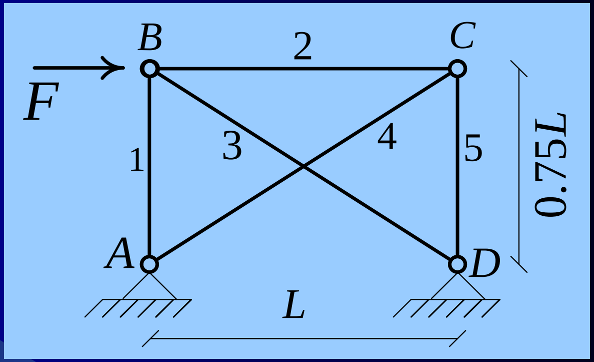
$$F_0 = \frac{M_0}{2L}$$

Displacement at Incipient Collapse in Limit Analysis, Example:

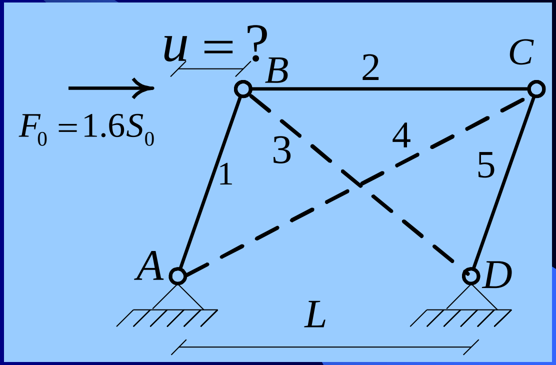
$$A_1 = A_2 = A_3 = A_4 = A_5 = A$$

$$\sigma_{y1} = \sigma_{y2} = \sigma_{y3} = \sigma_{y4} = \sigma_{y5} = \sigma_0$$

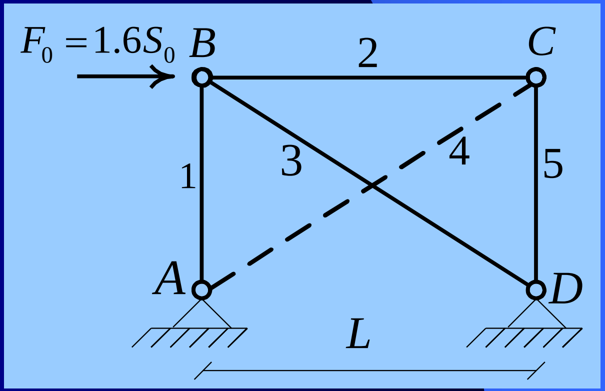
$$\sigma_0 A = S_0$$



-The actual collapse mechanism of the truss:



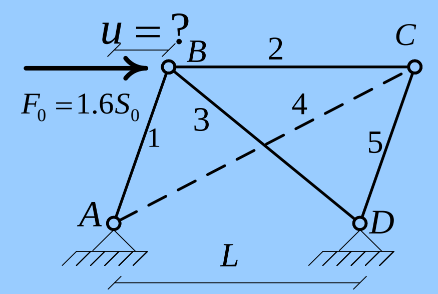
Alternative 1: Suppose that bar 3 yields last. Then at incipient collapse bar 3 is elastic yet.



Displacement at Incipient Collapse in Limit Analysis, Example, continue…:

-Defining the displacement u using virtual work:

Actual displacement:



Both of equilibrium and compatibility is needed.

$$\begin{cases} S_1 = 0.6S_0 \\ S_2 = -0.8S_0 \\ S_3 = -S_0 \\ S_4 = S_0 \\ S_5 = -0.6S_0 \end{cases}$$

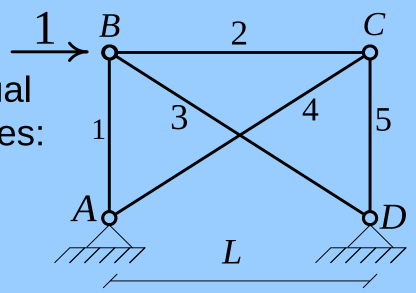
$$\begin{cases} e_1 = \dfrac{0.6S_0 \times 0.75L}{EA} = 0.45\dfrac{S_0 L}{EA} \\ e_2 = \dfrac{-0.8S_0 \times L}{EA} = -0.8\dfrac{S_0 L}{EA} \\ e_3 = \dfrac{-S_0 \times 1.25L}{EA} = -1.25\dfrac{S_0 L}{EA} \\ e_4 = \dfrac{S_0 \times 1.25L}{EA} + e_{4,p} = ? \\ e_5 = \dfrac{-0.6S_0 \times 0.75L}{EA} = -0.45\dfrac{S_0 L}{EA} \end{cases}$$

Displacement at Incipient Collapse in Limit Analysis, Example, continue…:

-Define the displacement u using virtual work, continue…:

$$\begin{cases} \delta S_1 = 0.75 \\ \delta S_2 = 0 \\ \delta S_3 = -1.25 \\ \delta S_4 = 0 \leftarrow suppose \\ \delta S_5 = 0 \end{cases}$$

Virtual forces:



Only equilibrium between internal and external forces is needed.

$$1 \times u = \sum_i \delta S_i \times e_i$$

$$1 \times u = 0.75 \times 0.45 \frac{S_0 L}{EA} - 1.25 \times \left(-1.25 \frac{S_0 L}{EA} \right)$$

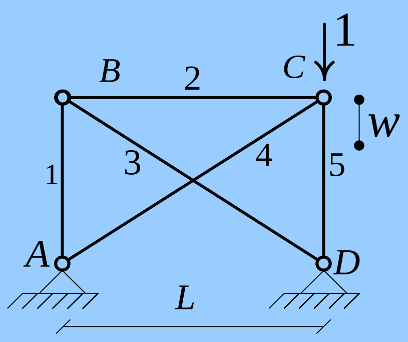
$$u = 1.9 \frac{S_0 L}{EA}$$

Displacement at Incipient Collapse in Limit Analysis, Example, continue…:

-Define the displacement w using virtual work, continue…:

$$\begin{cases} \delta S_1 = 0 \\ \delta S_2 = 0 \\ \delta S_3 = 0 \\ \delta S_4 = 0 \leftarrow suppose \\ \delta S_5 = -1 \end{cases}$$

Virtual forces:



Only equilibrium between internal and external forces is needed.

$$1 \times w = \sum_i \delta S_i \times e_i$$

$$1 \times w = -1 \times \left(-0.45 \frac{S_0 L}{EA} \right)$$

$$w = 0.45 \frac{S_0 L}{EA}$$

$$e_4 = 0.8(u + e_2) - 0.6w$$

This is impossible.

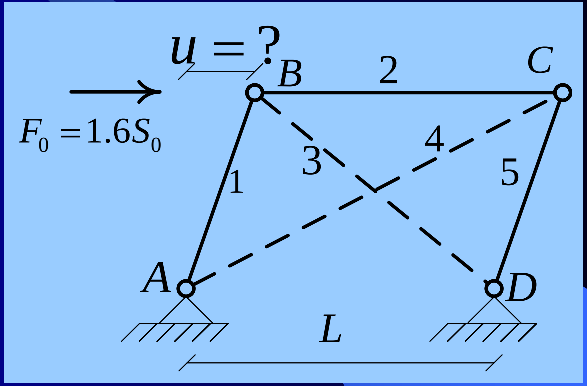
$$e_4 = 0.8 \times (1.9 - 0.8) \frac{S_0 L}{EA} - 0.6 \times \left(0.45 \frac{S_0 L}{EA} \right)$$

$$e_4 = 0.61 \frac{S_0 L}{EA}$$

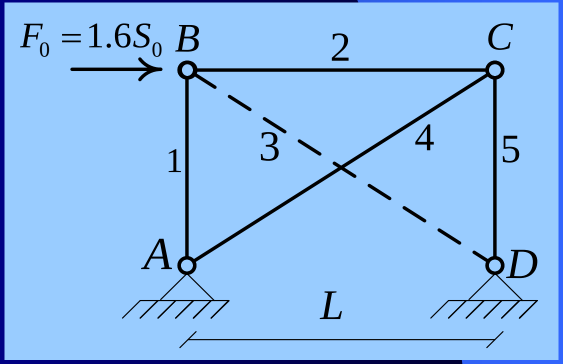
$$|e_4| < e_{4,e}^{\max} = 1.25 \frac{S_0 L}{EA}$$

Displacement at Incipient Collapse in Limit Analysis, Example, continue…:

-The actual collapse mechanism of the truss:

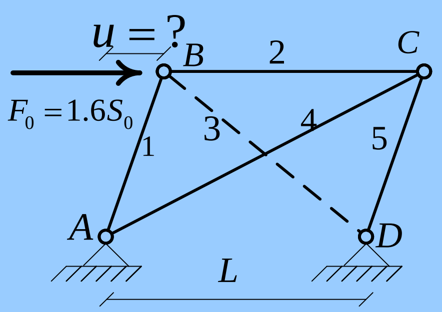


Alternative 2: Suppose that bar 4 yields last. Then at incipient collapse bar 4 is elastic yet.



Displacement at Incipient Collapse in Limit Analysis, Example, continue…:

Actual displacement:



Both of equilibrium and compatibility is needed.

-Defining the displacement u using virtual work:

$$\begin{cases} S_1 = 0.6S_0 \\ S_2 = -0.8S_0 \\ S_3 = -S_0 \\ S_4 = S_0 \\ S_5 = -0.6S_0 \end{cases}$$

$$\Longrightarrow$$

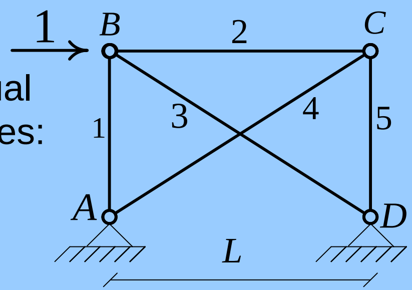
$$\begin{cases} e_1 = \dfrac{0.6S_0 \times 0.75L}{EA} = 0.45\dfrac{S_0L}{EA} \\ e_2 = \dfrac{0.8S_0 \times L}{EA} = -0.8\dfrac{S_0L}{EA} \\ e_3 = \dfrac{-S_0 \times 1.25L}{EA} + e_{3,p} = ? \\ e_4 = \dfrac{S_0 \times 1.25L}{EA} = 1.25\dfrac{S_0L}{EA} \\ e_5 = \dfrac{-0.6S_0 \times 0.75L}{EA} = -0.45\dfrac{S_0L}{EA} \end{cases}$$

Displacement at Incipient Collapse in Limit Analysis, Example, continue…:

-Define the displacement u using virtual work, continue…:

$$\begin{cases} \delta S_1 = 0 \\ \delta S_2 = -1 \\ \delta S_3 = 0 \leftarrow suppose \\ \delta S_4 = 1.25 \\ \delta S_5 = -0.75 \end{cases}$$

Virtual forces:



Only equilibrium between internal and external forces is needed.

$$1 \times u = \sum_i \delta S_i \times e_i$$

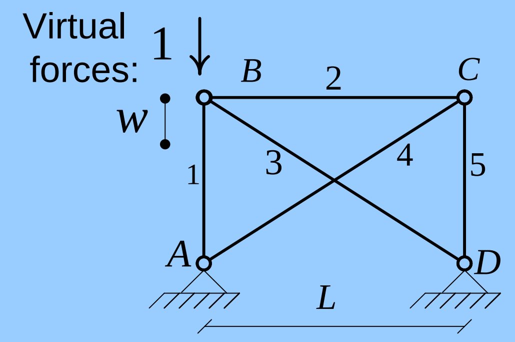
$$1 \times u = -1 \times \left(-0.8 \frac{S_0 L}{EA} \right) + 1.25 \times 1.25 \frac{S_0 L}{EA} - 0.75 \times \left(-0.45 \frac{S_0 L}{EA} \right)$$

$$u = 2.7 \frac{S_0 L}{EA}$$

Displacement at Incipient Collapse in Limit Analysis, Example, continue…:

-Define the displacement w using virtual work, continue…:

$$\begin{cases} \delta S_1 = -1 \\ \delta S_2 = 0 \\ \delta S_3 = 0 \leftarrow \textit{suppose} \\ \delta S_4 = 0 \\ \delta S_5 = 0 \end{cases}$$



Only equilibrium between internal and external forces is needed.

$$1 \times w = \sum_i \delta S_i \times e_i$$

$$1 \times w = -1 \times 0.45 \frac{S_0 L}{EA}$$

$$w = -0.45 \frac{S_0 L}{EA}$$

$$e_3 = -0.8u - 0.6w$$

This is the true condition.

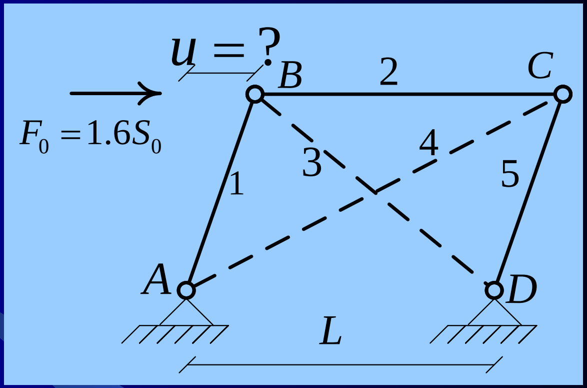
$$e_3 = -0.8 \times 2.7 \frac{S_0 L}{EA} - 0.6 \times \left(-0.45 \frac{S_0 L}{EA} \right)$$

$$e_3 = -1.89 \frac{S_0 L}{EA}$$

$$|e_3| > e_{3,e}^{\max} = 1.25 \frac{S_0 L}{EA}$$

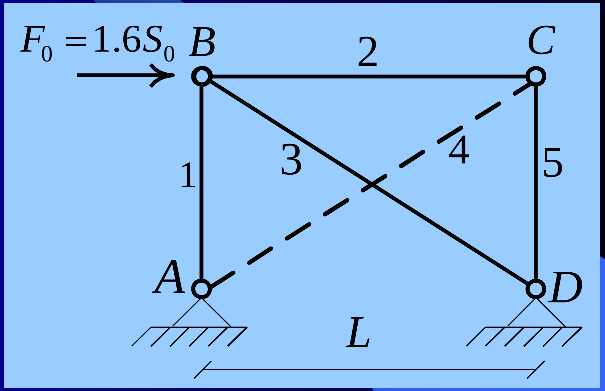
Displacement at Incipient Collapse in Limit Analysis, Example, continue…:

-The actual collapse mechanism of the truss:



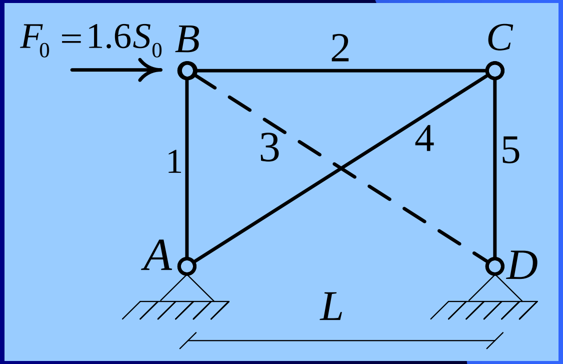
Alternative 1 (impossible): The bar 3 yields last. Then at incipient collapse bar 3 is elastic yet.

$$\boxed{u = 1.9\frac{S_0 L}{EA}}$$



Alternative 2 (Actual): The bar 4 yields last. Then at incipient collapse bar 4 is elastic yet.

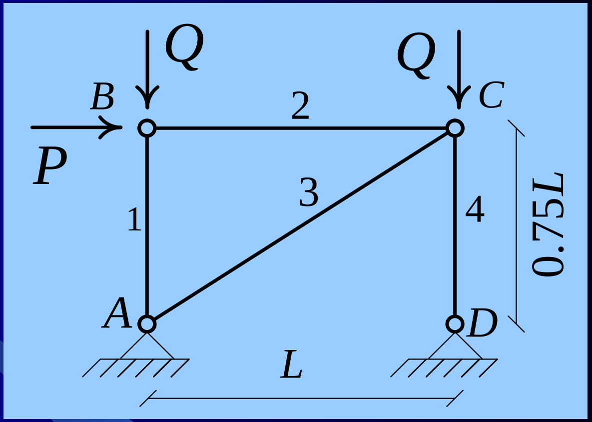
$$\boxed{u = 2.7\frac{S_0 L}{EA}}$$



Displacement at Incipient Collapse in Limit Analysis, continue…:

-The actual displacement maximizes the work done by the reference loading, provided that no bar has been unloaded after yielding.

-Safe Domain of Nonproportional Loading in Limit Analysis, example:



$$A_1 = A_2 = A_3 = A_4 = A$$

$$\sigma^t_{y1} = \sigma^t_{y2} = \sigma^t_{y3} = \sigma^t_{y4} = \sigma_0$$

$$S^t_0 = \sigma_0 A = S_0$$

$$\sigma^c_{y1} = \sigma^c_{y2} = \sigma^c_{y3} = \sigma^c_{y4} = 0.7\sigma_0$$

$$S^c_0 = 0.7\sigma_0 A = 0.7 S_0$$

- The number of all potential mechanisms:

$$\binom{4}{4} = \frac{(4!)}{(1!)(3!)} = 4$$

- The number of bars: m=4

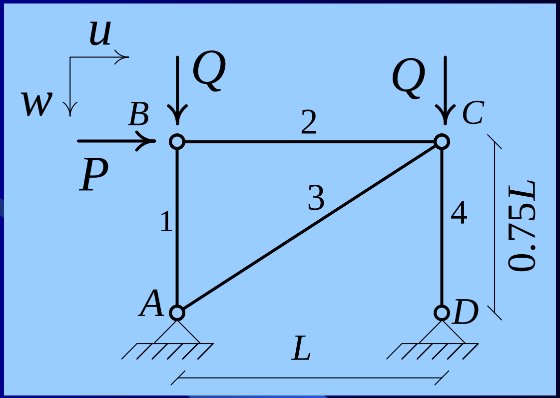
- The degree of freedom: n=4

- The degree of static redundancy: s = m – n = 0

-Safe Domain of Nonproportional Loading in Limit Analysis, example, continue…:

- The kinematic equations:

$$\begin{cases} \dot{e}_1 = -\dot{w}_B \\ \dot{e}_2 = \dot{u}_C - \dot{u}_B \\ \dot{e}_3 = 0.8\dot{u}_C - 0.6\dot{w}_C \\ \dot{e}_4 = -\dot{w}_C \end{cases}$$

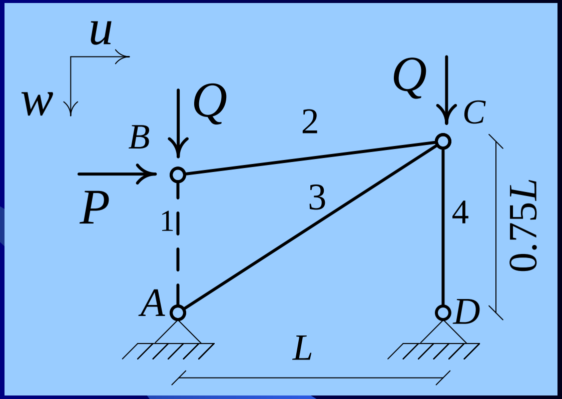


- The equilibrium equations:

$$\begin{cases} S_1 = -Q \\ S_2 = -P \\ S_2 + 0.8S_3 = 0 \\ S_4 + 0.6S_3 = -Q \end{cases}$$

-Safe Domain of Nonproportional Loading in Limit Analysis, example, continue…:

- Mechanism 1:



$$D_{\text{int}} \geq \dot{W}_{ext} \Longrightarrow S^{c}_{01}|\dot{e}_1| \geq Q\dot{w}_B$$

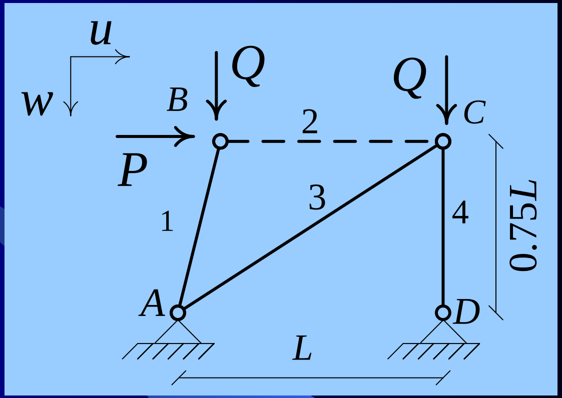
$$0.7S_0 \times \dot{w}_B \geq Q\dot{w}_B \Longrightarrow Q \leq 0.7S_0$$

$$D_{\text{int}} \geq \dot{W}_{ext} \Longrightarrow S^{t}_{01}|\dot{e}_1| \geq -Q\dot{w}_B$$

$$S_0 \times \dot{w}_B \geq -Q\dot{w}_B \Longrightarrow Q \geq -S_0$$

-Safe Domain of Nonproportional Loading in Limit Analysis, example, continue…:

- Mechanism 2:



$D_{\text{int}} \geq \dot{W}_{ext} \quad \Longrightarrow \quad S^{c}_{02} |\dot{e}_2| \geq P\dot{u}_B$

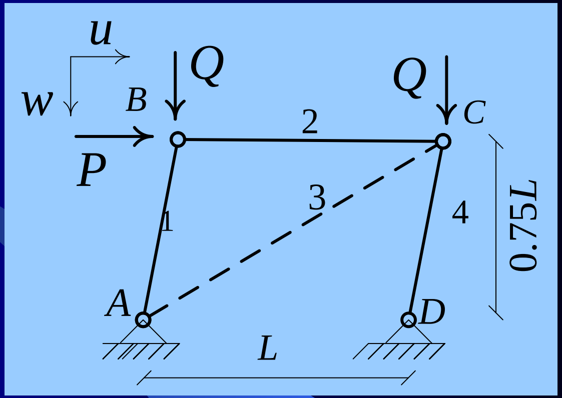
$0.7S_0 \times \dot{u}_B \geq P\dot{u}_B \quad \Longrightarrow \quad P \leq 0.7S_0$

$D_{\text{int}} \geq \dot{W}_{ext} \quad \Longrightarrow \quad S^{t}_{02} |\dot{e}_2| \geq -P\dot{u}_B$

$S_0 \times \dot{u}_B \geq -P\dot{u}_B \quad \Longrightarrow \quad P \geq -S_0$

-Safe Domain of Nonproportional Loading in Limit Analysis, example, continue…:

- Mechanism 3:



$$D_{\text{int}} \geq \dot{W}_{ext} \Longrightarrow S_{03}^{t}\left|\dot{e}_3\right| \geq P\dot{u}_B$$

$$S_0 \times 0.8\dot{u}_B \geq P\dot{u}_B \Longrightarrow P \leq 0.8 S_0$$

$$D_{\text{int}} \geq \dot{W}_{ext} \Longrightarrow S_{03}^{c}\left|\dot{e}_3\right| \geq -P\dot{u}_B$$

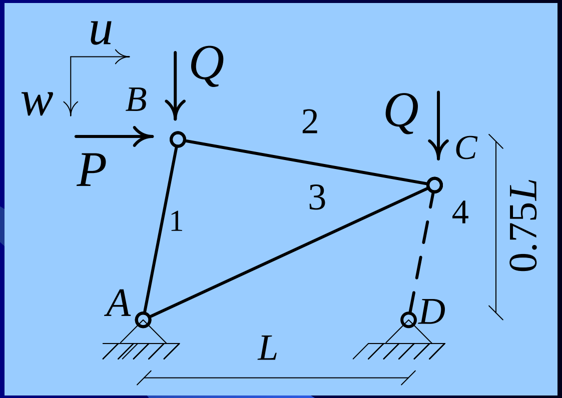
$$0.7S_0 \times 0.8\dot{u}_B \geq -P\dot{u}_B \Longrightarrow P \geq -0.56 S_0$$

-Safe Domain of Nonproportional Loading in Limit Analysis, example, continue…:

- Mechanism 4:

$$\dot{e}_2 = \dot{u}_C - \dot{u}_B = 0 \rightarrow \dot{u}_C = \dot{u}_B$$

$$\dot{e}_3 = 0.8\dot{u}_C - 0.6\dot{w}_C = 0 \rightarrow \dot{w}_C = \frac{4}{3}\dot{u}_C$$



$$D_{\text{int}} \geq \dot{W}_{ext} \Rightarrow S_{04}^{c}\left|\dot{e}_4\right| \geq P\dot{u}_B + Q\dot{w}_C$$

$$0.7S_0 \times \dot{w}_C \geq P\dot{u}_B + Q\dot{w}_C \Rightarrow 0.7S_0 \times \frac{4}{3}\dot{u}_B \geq P\dot{u}_B + Q\frac{4}{3}\dot{u}_B \qquad 3P + 4Q \leq 2.8S_0$$

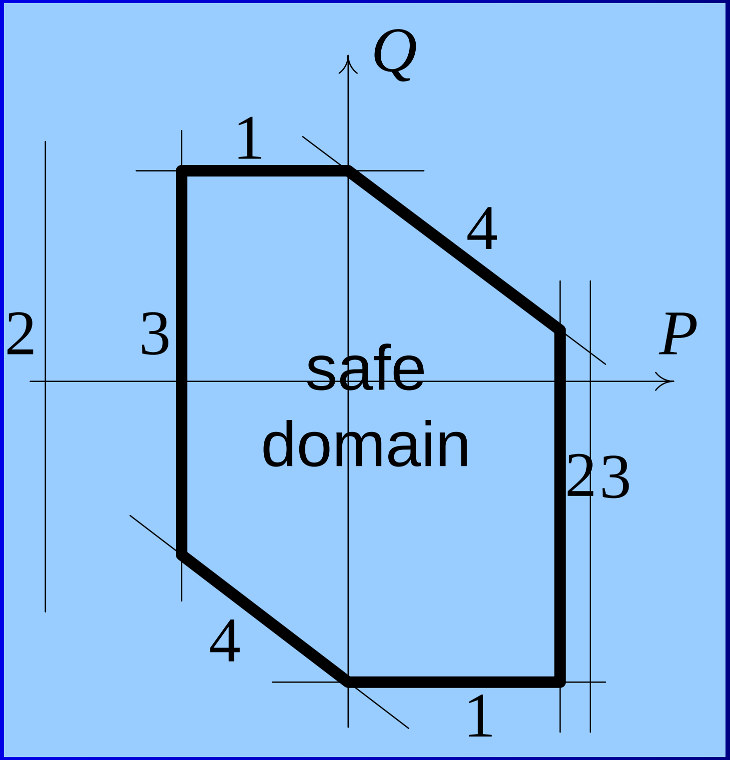
$$D_{\text{int}} \geq \dot{W}_{ext} \Rightarrow S_{04}^{t}\left|\dot{e}_4\right| \geq -P\dot{u}_B - Q\dot{w}_C$$

$$S_0 \times \dot{w}_C \geq -P\dot{u}_B - Q\dot{w}_C \Rightarrow S_0 \times \frac{4}{3}\dot{u}_B \geq -P\dot{u}_B - Q\frac{4}{3}\dot{u}_B \qquad 3P + 4Q \geq -4S_0$$

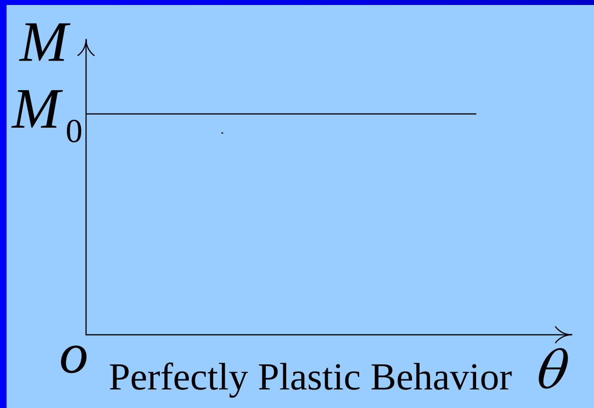
-Safe Domain of Nonproportional Loading in Limit Analysis, example, continue…:

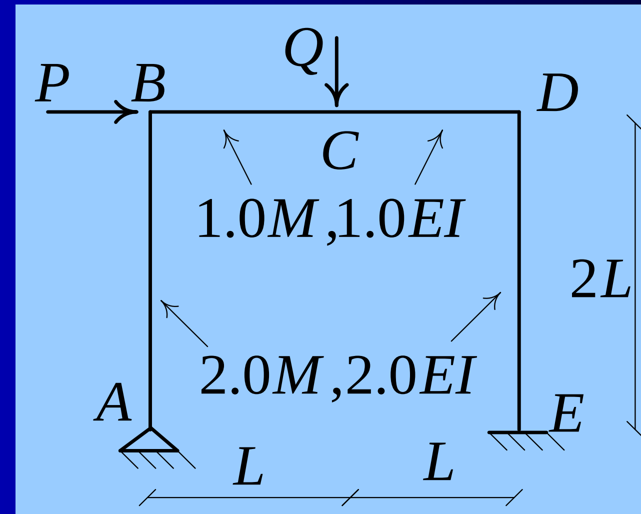
$Q \le 0.7S_0$ $\quad$ $P \le 0.7S_0$ $\quad$ $P \le 0.8S_0$ $\quad$ $3P + 4Q \le 2.8S_0$

$Q \ge -S_0$ $\quad$ $P \ge -S_0$ $\quad$ $P \ge -0.56S_0$ $\quad$ $3P + 4Q \ge -4S_0$



Safe Domain, Example:





-The frame is statically indeterminate to second degree and has four critical sections as B, C, D and E, so collapse occurs if the hinges are inserted in these sections.

-Number of potential mechanisms:

$$\binom{4}{3} = \frac{(4!)}{(3!)(1!)} = 4$$

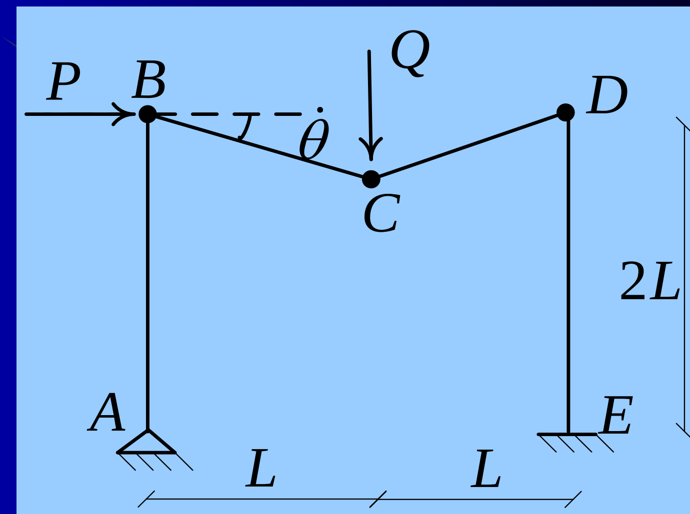
Safe Domain, Example, continue…:

-Alternative 1:

$$D_{\text{int}} \geq \dot{W}_{ext}$$

$$M_0\left|\dot{\theta}\right| + M_0\left|2\dot{\theta}\right| + M_0\left|\dot{\theta}\right| \geq Q\dot{\theta}L$$

$$-\frac{4M_0}{L} \leq Q \leq \frac{4M_0}{L}$$

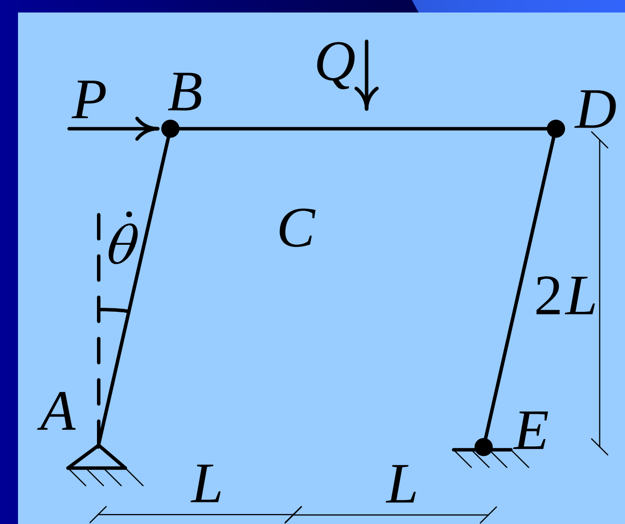


-Alternative 2:

$$D_{\text{int}} \geq \dot{W}_{ext}$$

$$M_0\left|\dot{\theta}\right| + M_0\left|\dot{\theta}\right| + 2M_0\left|\dot{\theta}\right| \geq P\dot{\theta}2L$$

$$-\frac{2M_0}{L} \leq P \leq \frac{2M_0}{L}$$



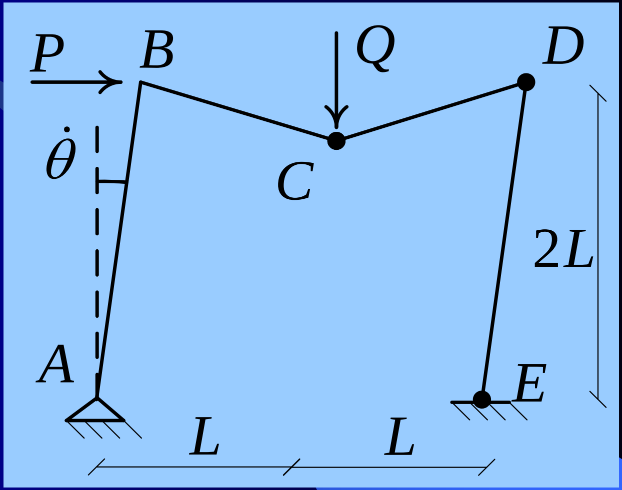
Safe Domain, Example, continue…:

-Alternative 3:

$$D_{\text{int}} \geq \dot{W}_{ext}$$

$$M_0\left|2\dot{\theta}\right| + M_0\left|2\dot{\theta}\right| + 2M_0\left|\dot{\theta}\right| \geq P\dot{\theta}2L + Q\dot{\theta}L$$

$$-\frac{6M_0}{L} \leq 2P + Q \leq \frac{6M_0}{L}$$

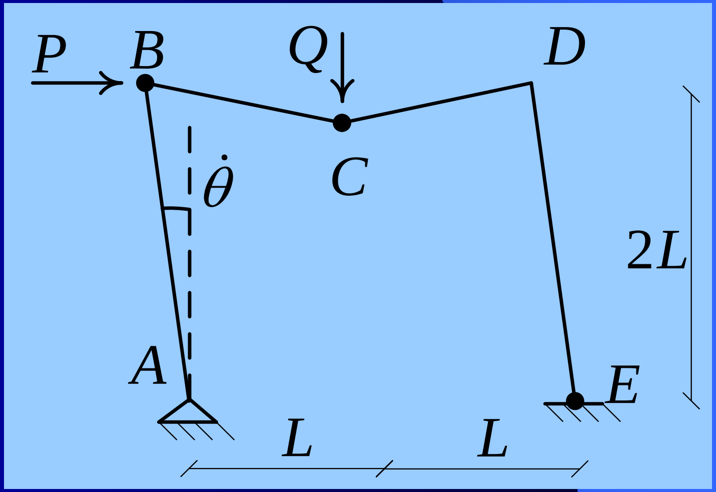


-Alternative 4:

$$D_{\text{int}} \geq \dot{W}_{ext}$$

$$M_0\left|2\dot{\theta}\right| + M_0\left|2\dot{\theta}\right| + 2M_0\left|\dot{\theta}\right| \geq -P\dot{\theta}2L + Q\dot{\theta}L$$

$$-\frac{6M_0}{L} \leq -2P + Q \leq \frac{6M_0}{6L}$$



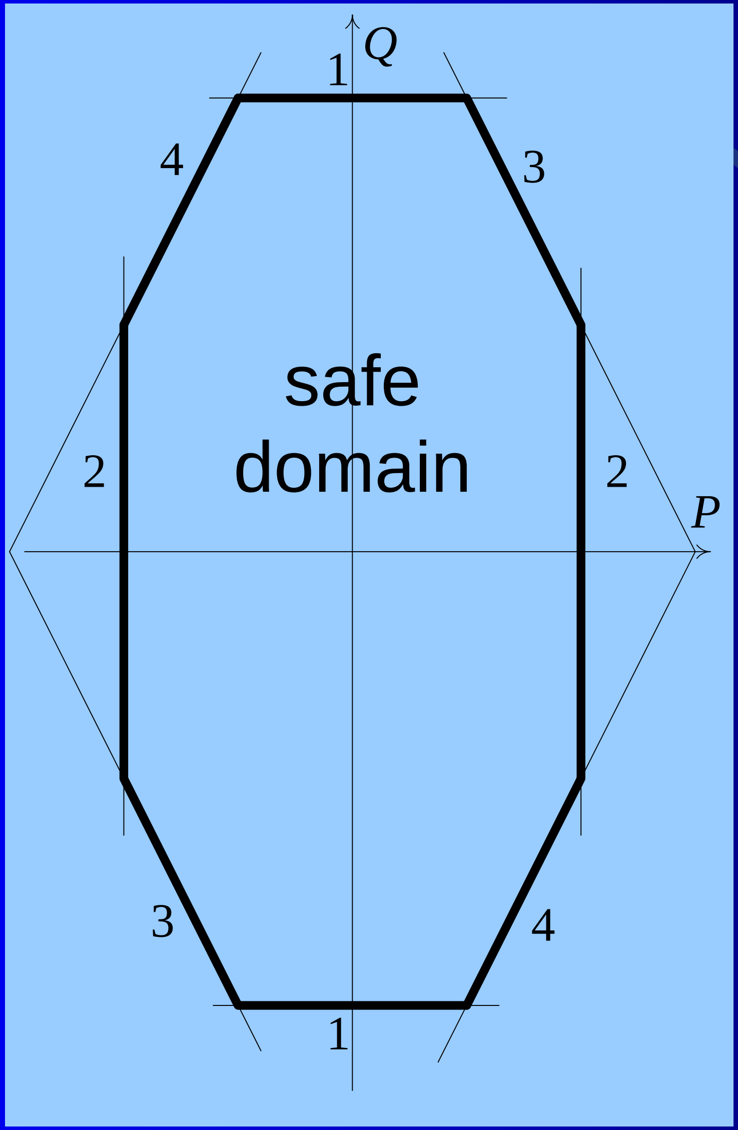
-Safe Domain, example, continue…:

$$-\frac{4M_0}{L} \le Q \le \frac{4M_0}{L}$$

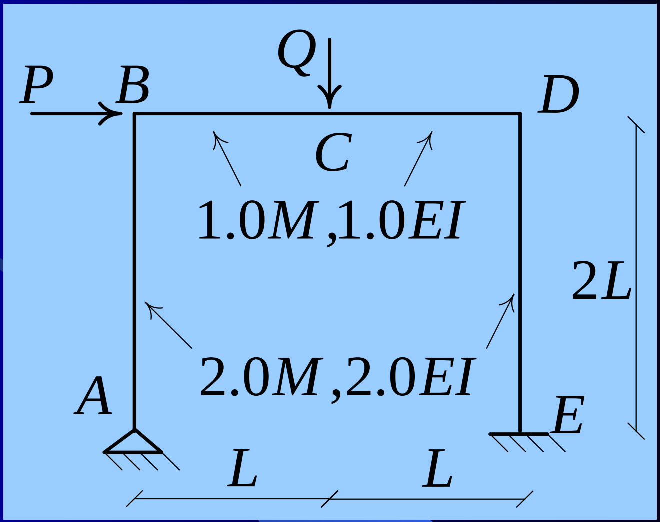
$$-\frac{2M_0}{L} \le P \le \frac{2M_0}{L}$$

$$-\frac{6M_0}{L} \le 2P + Q \le \frac{6M_0}{L}$$

$$-\frac{6M_0}{L} \le -2P + Q \le \frac{6M_0}{L}$$

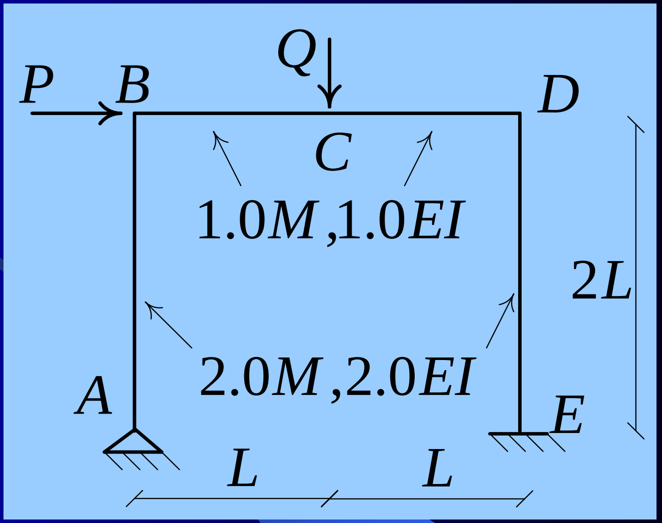


Elastic Domain, Example:



-The frame is statically indeterminate to second degree and has four critical sections as B, C, D and E, so elastic domain forms if one of these sections yields.

Elastic Domain, Example, Continue…:



-Elastic solution:

$$M_{BA} = M_{BC} = 0.43038PL - 0.20886QL$$

$$-M_0 \le 0.43038PL - 0.20886QL \le M_0$$

$$M_{CB} = M_{CD} = -0.06329PL + 0.31013QL$$

$$-M_0 \le -0.06329PL + 0.31013QL \le M_0$$

$$M_{DC} = M_{DE} = -0.55696PL - 0.17089QL$$

$$-M_0 \le -0.55696PL - 0.17089QL \le M_0$$

$$M_{ED} = 1.01266PL + 0.03797QL$$

$$-2M_0 \le 1.01266PL + 0.03797QL \le 2M_0$$

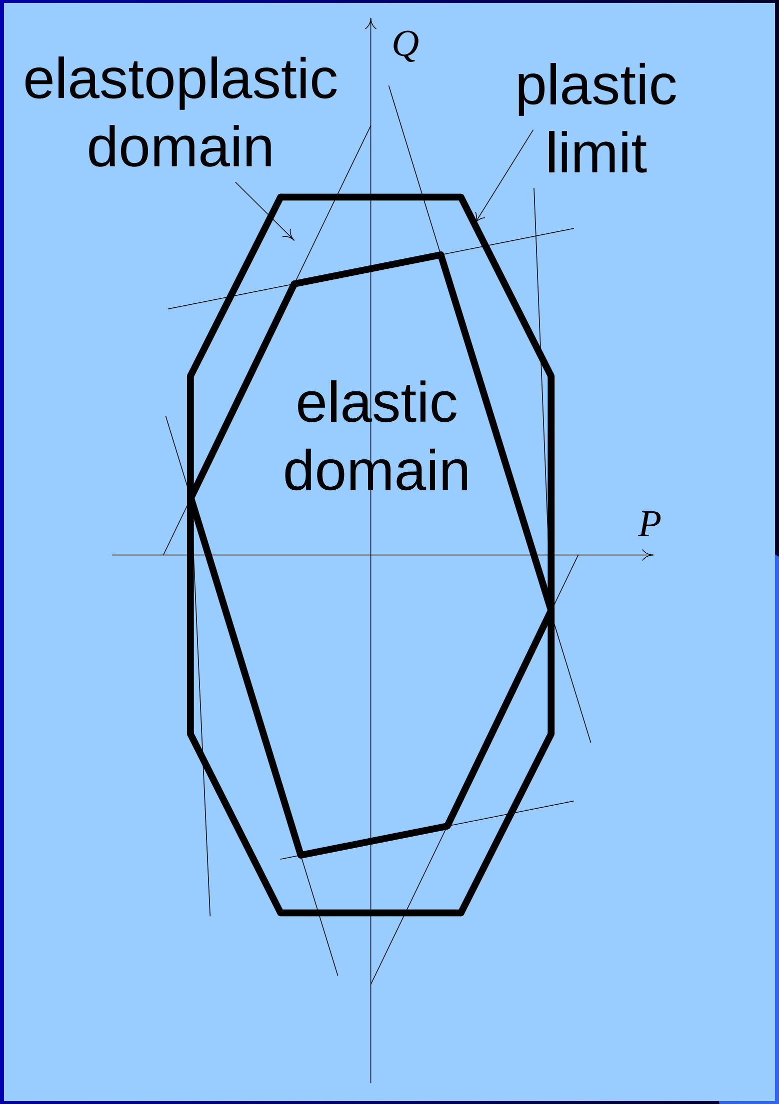
-Elastic Domain, example, continue…:

$$-M_0 \le 0.43038PL - 0.20886QL \le M_0$$

$$-M_0 \le -0.06329PL + 0.31013QL \le M_0$$

$$-M_0 \le -0.55696PL - 0.17089QL \le M_0$$

$$-2M_0 \le 1.01266PL + 0.03797QL \le 2M_0$$



-Cyclic Loading, example:

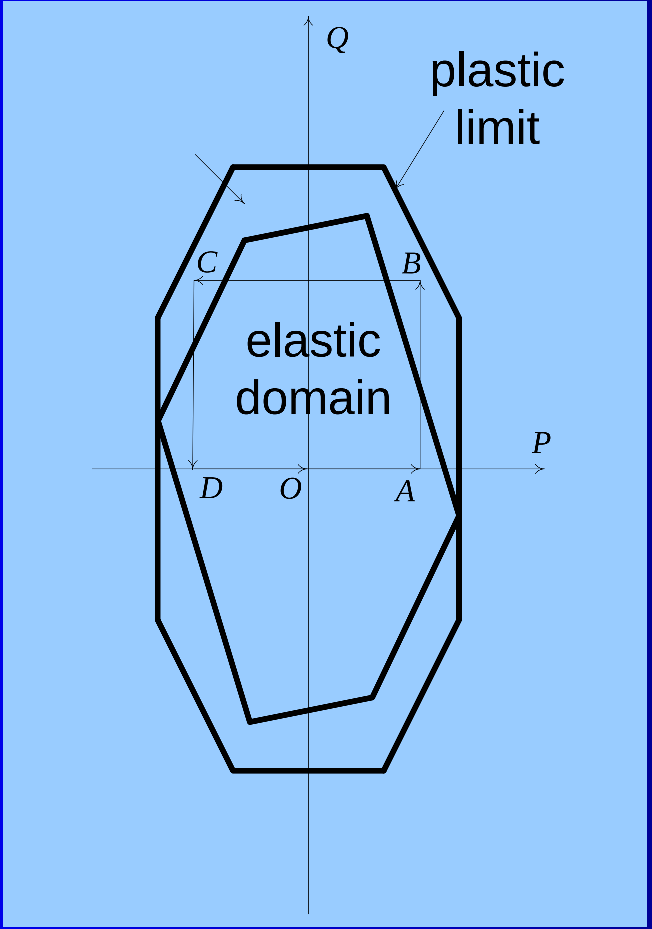
$$O:\begin{cases} P=0 \\ Q=0 \end{cases}$$

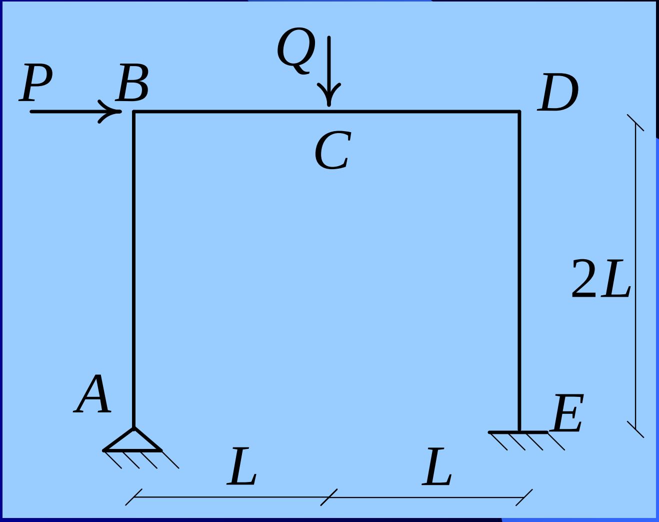
$$D:\begin{cases} P=-\dfrac{1.5M_0}{L} \\ Q=0 \end{cases}$$

$$A:\begin{cases} P=\dfrac{1.5M_0}{L} \\ Q=0 \end{cases}$$

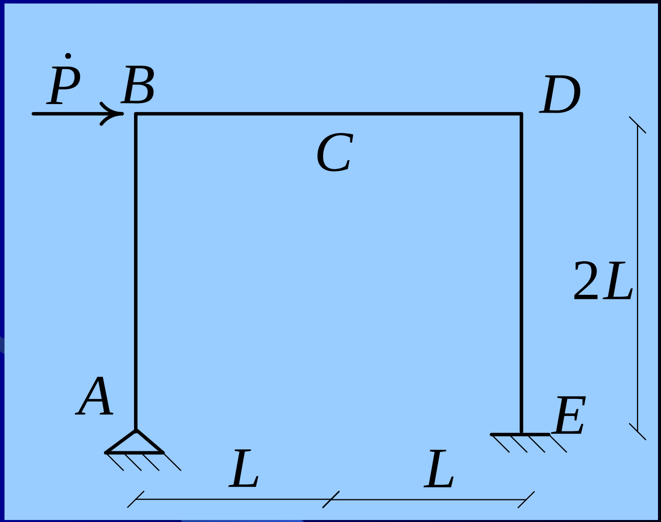
$$B:\begin{cases} P=\dfrac{1.5M_0}{L} \\ Q=\dfrac{2.5M_0}{L} \end{cases}$$

$$C:\begin{cases} P=-\dfrac{1.5M_0}{L} \\ Q=\dfrac{2.5M_0}{L} \end{cases}$$





Step 1:



$$\dot{M}_{BA} = \dot{M}_{BC} = 0.43038\dot{P}L$$

$$\dot{M}_{CB} = \dot{M}_{CD} = -0.06329\dot{P}L$$

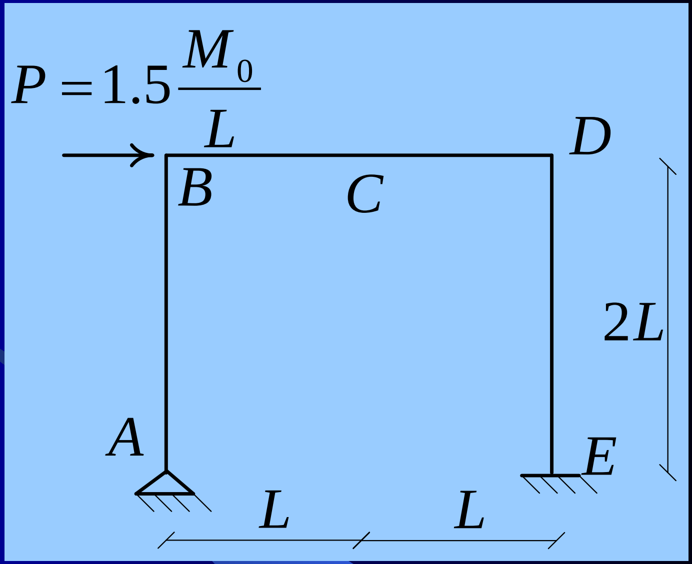
$$\dot{M}_{DC} = \dot{M}_{DE} = -0.55696\dot{P}L$$

$$\dot{M}_{ED} = 1.01266\dot{P}L$$

$$\dot{u}_B = \left(489.45\frac{\dot{P}L^3}{EI}\right)\times 10^{-3}$$

$$\dot{w}_C = \left(-31.65\frac{\dot{P}L^3}{EI}\right)\times 10^{-3}$$

Step 1, Loading A:



$$\dot{P} = 1.5\frac{M_0}{L}$$

$$M_{BA} = M_{BC} = 0.64557M_0$$

$$M_{CB} = M_{CD} = -0.09494M_0$$

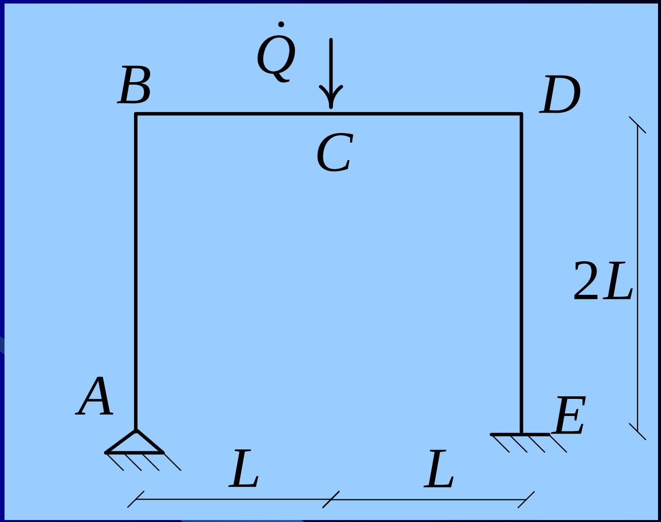
$$M_{DC} = M_{DE} = -0.83544M_0$$

$$M_{ED} = 1.51899M_0$$

$$u_B = 734.18\times10^{-3}\frac{M_0L^2}{EI}$$

$$w_C = -47.48\times10^{-3}\frac{M_0L^2}{EI}$$

Step 2:



$$\dot{M}_{BA} = \dot{M}_{BC} = -0.20886\dot{Q}L$$

$$\dot{M}_{CB} = \dot{M}_{CD} = 0.31013\dot{Q}L$$

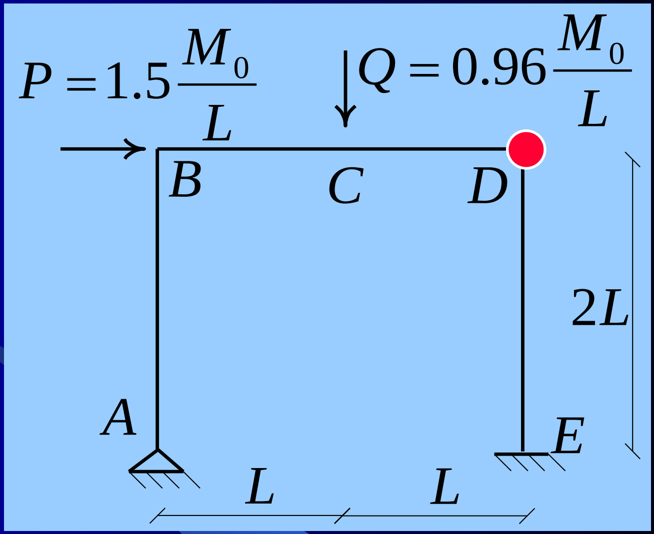
$$\dot{M}_{DC} = \dot{M}_{DE} = -0.17089\dot{Q}L$$

$$\dot{M}_{ED} = 0.03797\dot{Q}L$$

$$\dot{u}_B = \left(-31.65\frac{\dot{P}L^3}{EI} \right) \times 10^{-3}$$

$$\dot{w}_C = \left(71.73\frac{\dot{Q}L^3}{EI} \right) \times 10^{-3}$$

Step 2, Loading between A and B:



$$\dot{Q} = 0.96\frac{M_0}{L}$$

$$M_{BA} = M_{BC} = 0.44506 M_0$$

$$M_{CB} = M_{CD} = 0.20278 M_0$$

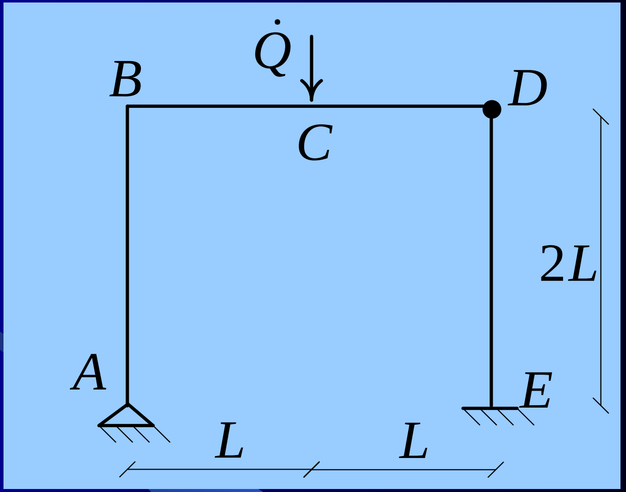
$$M_{DC} = M_{DE} = -M_0$$

$$M_{ED} = 1.55544 M_0$$

$$u_B = 703.80\times10^{-3}\frac{M_0 L^2}{EI}$$

$$w_C = 21.38\times10^{-3}\frac{M_0 L^2}{EI}$$

Step 3:



$$\dot{M}_{BA} = \dot{M}_{BC} = -0.18750\dot{Q}L$$

$$\dot{M}_{CB} = \dot{M}_{CD} = 0.40625\dot{Q}L$$

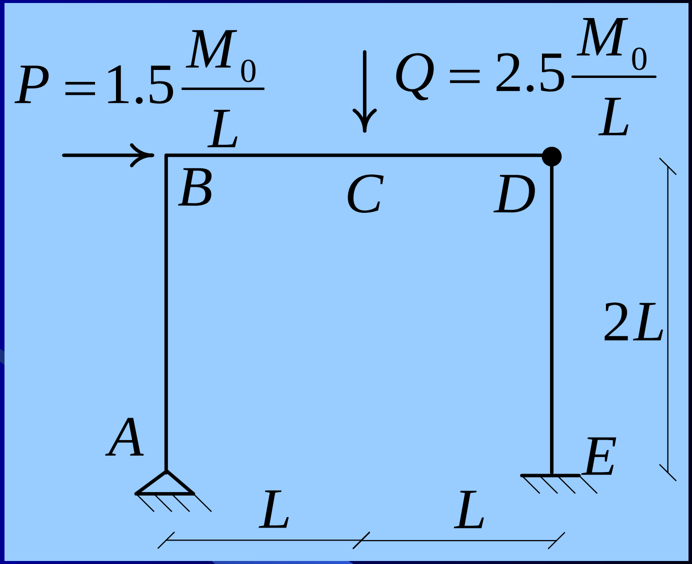
$$\dot{M}_{DC} = \dot{M}_{DE} = 0$$

$$\dot{M}_{ED} = 0.18750\dot{Q}L$$

$$\dot{u}_B = \left(125.00\frac{\dot{Q}L^3}{EI}\right)\times 10^{-3}$$

$$\dot{w}_C = \left(119.79\frac{\dot{Q}L^3}{EI}\right)\times 10^{-3}$$

Step 3, Loading B:



$$\dot{Q} = 1.54\frac{M_0}{L}$$

$$M_{BA} = M_{BC} = 0.15631M_0$$

$$M_{CB} = M_{CD} = 0.82841M_0$$

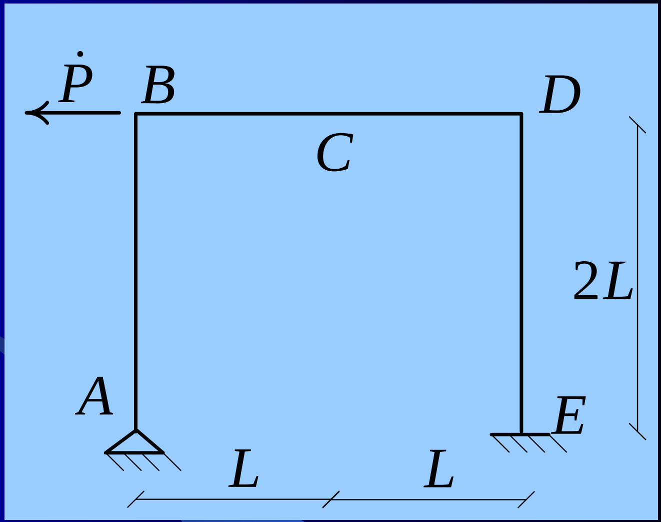
$$M_{DC} = M_{DE} = -M_0$$

$$M_{ED} = 1.84419M_0$$

$$u_B = 896.30\times10^{-3}\frac{M_0L^2}{EI}$$

$$w_C = 205.86\times10^{-3}\frac{M_0L^2}{EI}$$

Step 4:



$$\dot{M}_{BA} = \dot{M}_{BC} = -0.43038\dot{P}L$$

$$\dot{M}_{CB} = \dot{M}_{CD} = 0.06329\dot{P}L$$

$$\dot{M}_{DC} = \dot{M}_{DE} = 0.55696\dot{P}L$$

$$\dot{M}_{ED} = -1.01266\dot{P}L$$

$$\dot{u}_B = \left(-489.45\frac{\dot{P}L^3}{EI}\right)\times 10^{-3}$$

$$\dot{w}_C = \left(31.65\frac{\dot{P}L^3}{EI}\right)\times 10^{-3}$$

Step 4, Loading between B and C:

$$\dot{P} = 2.68672\frac{M_0}{L}$$

$$M_{BA} = M_{BC} = -M_0$$

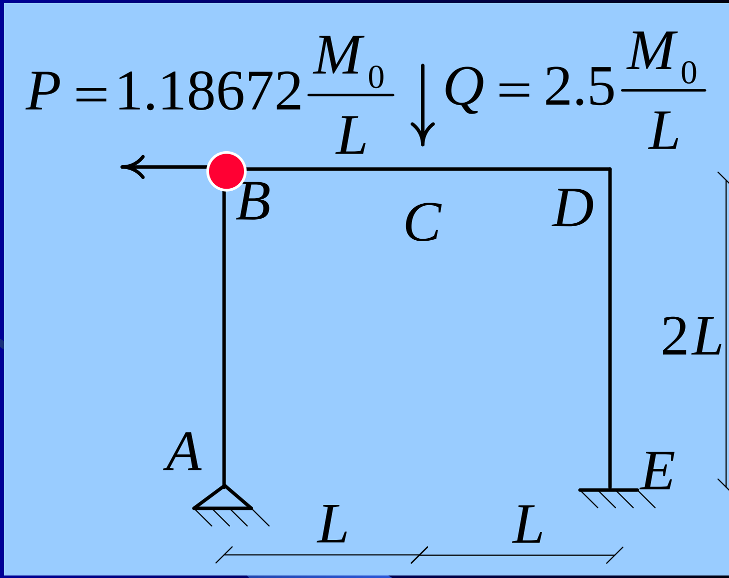
$$M_{CB} = M_{CD} = 0.99845M_0$$

$$M_{DC} = M_{DE} = 0.49640M_0$$

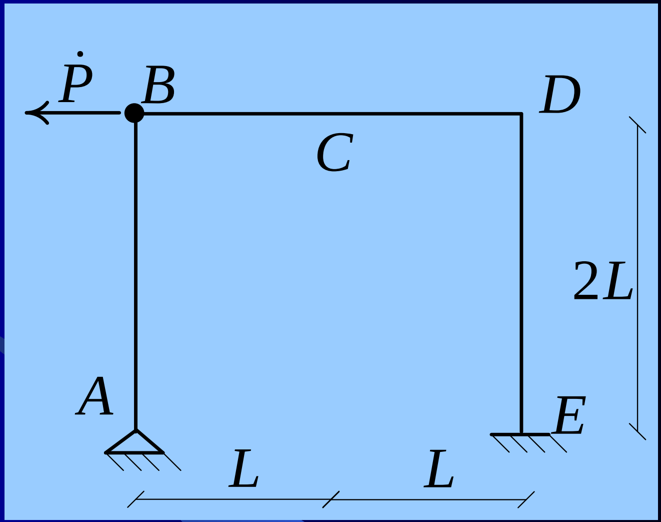
$$M_{ED} = -0.87654M_0$$

$$u_B = -418.72\times10^{-3}\frac{M_0L^2}{EI}$$

$$w_C = 290.89\times10^{-3}\frac{M_0L^2}{EI}$$



Step 5:



$$\dot{M}_{BA} = \dot{M}_{BC} = 0$$

$$\dot{M}_{CB} = \dot{M}_{CD} = 0.30000\dot{P}L$$

$$\dot{M}_{DC} = \dot{M}_{DE} = 0.60000\dot{P}L$$

$$\dot{M}_{ED} = -1.40000\dot{P}L$$

$$\dot{u}_B = \left(-733.33 \frac{\dot{P}L^3}{EI} \right) \times 10^{-3}$$

$$\dot{w}_C = \left(150.00 \frac{\dot{P}L^3}{EI} \right) \times 10^{-3}$$

Step 5, Loading between B and C:

$$\dot{P} = 0.00517\frac{M_0}{L}$$

$$M_{BA} = M_{BC} = -M_0$$

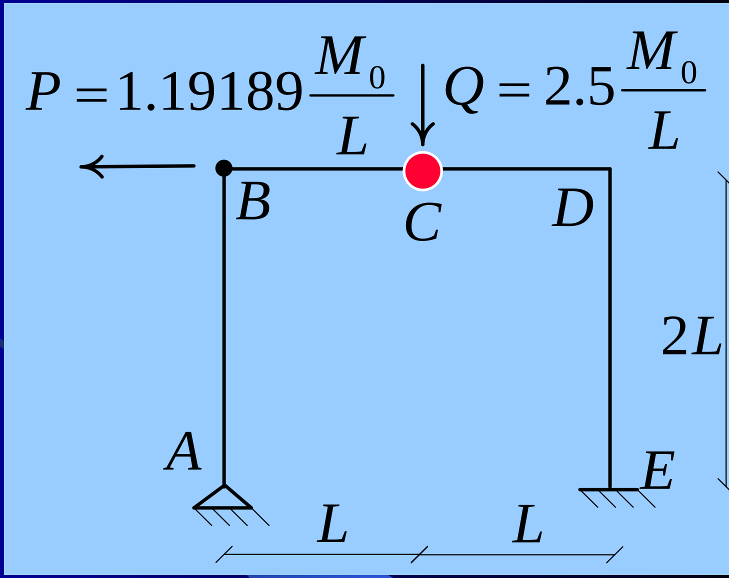
$$M_{CB} = M_{CD} = M_0$$

$$M_{DC} = M_{DE} = 0.49950M_0$$

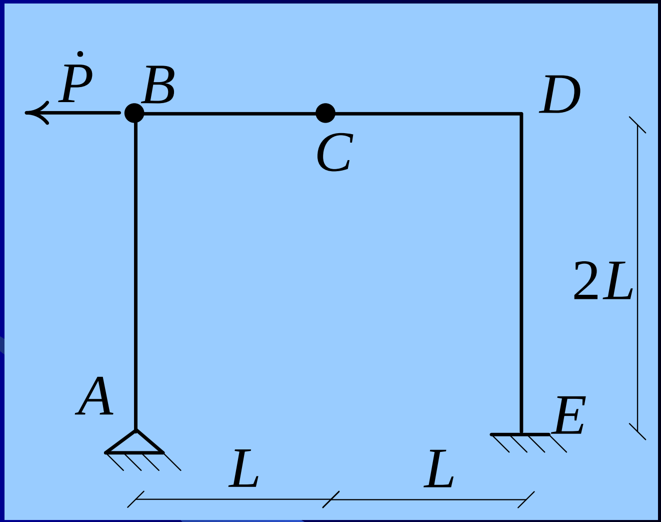
$$M_{ED} = -0.88378M_0$$

$$u_B = -422.51\times10^{-3}\frac{M_0L^2}{EI}$$

$$w_C = 291.67\times10^{-3}\frac{M_0L^2}{EI}$$



Step 6:



$$\dot{M}_{BA} = \dot{M}_{BC} = 0$$

$$\dot{M}_{CB} = \dot{M}_{CD} = 0$$

$$\dot{M}_{DC} = \dot{M}_{DE} = 0$$

$$\dot{M}_{ED} = -2\dot{P}L$$

$$\dot{u}_B = \left(-1333.33\frac{\dot{Q}L^3}{EI}\right)\times 10^{-3}$$

$$\dot{w}_C = \left(1000.00\frac{\dot{Q}L^3}{EI}\right)\times 10^{-3}$$

Step 6, Loading C:

$$\dot{P} = 0.30811\frac{M_0}{L}$$

$$M_{BA} = M_{BC} = -M_0$$

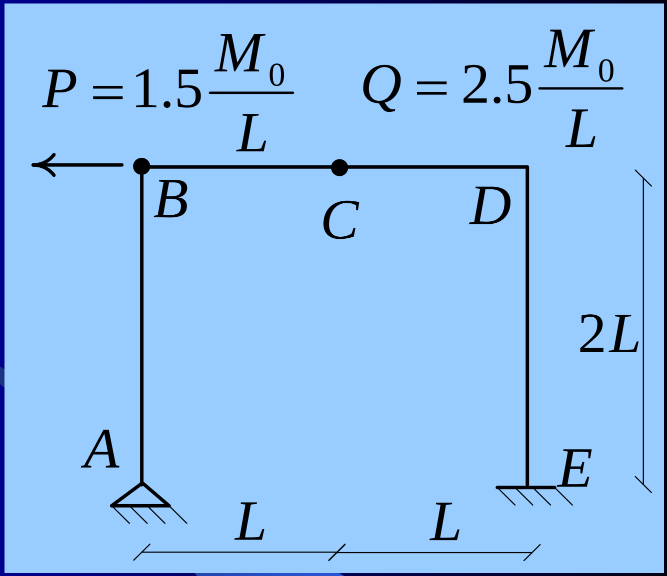
$$M_{CB} = M_{CD} = M_0$$

$$M_{DC} = M_{DE} = 0.49950M_0$$

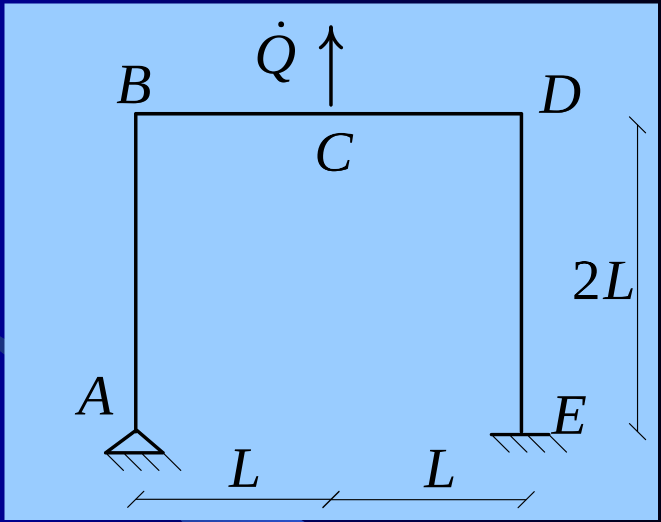
$$M_{ED} = -1.50000M_0$$

$$u_B = -833.32\times10^{-3}\frac{M_0L^2}{EI}$$

$$w_C = 599.78\times10^{-3}\frac{M_0L^2}{EI}$$



Step 7:



$$\dot{M}_{BA} = \dot{M}_{BC} = 0.20886\dot{Q}L$$

$$\dot{M}_{CB} = \dot{M}_{CD} = -0.31013\dot{Q}L$$

$$\dot{M}_{DC} = \dot{M}_{DE} = 0.17089\dot{Q}L$$

$$\dot{M}_{ED} = -0.03797\dot{Q}L$$

$$\dot{u}_B = \left(31.65\frac{\dot{P}L^3}{EI}\right)\times 10^{-3}$$

$$\dot{w}_C = \left(-71.73\frac{\dot{Q}L^3}{EI}\right)\times 10^{-3}$$

Step 7, Loading D:

$$\dot{Q} = 2.5\frac{M_0}{L}$$

$$M_{BA} = M_{BC} = -0.47785M_0$$

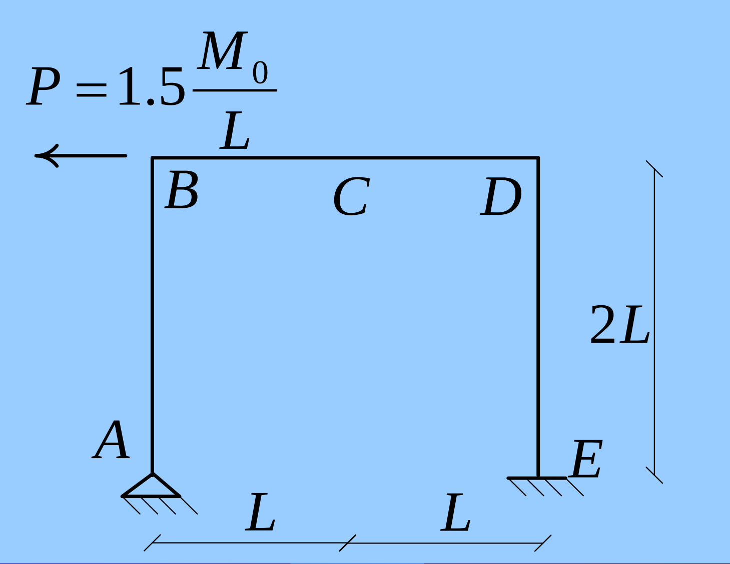
$$M_{CB} = M_{CD} = 0.22468M_0$$

$$M_{DC} = M_{DE} = 0.92673M_0$$

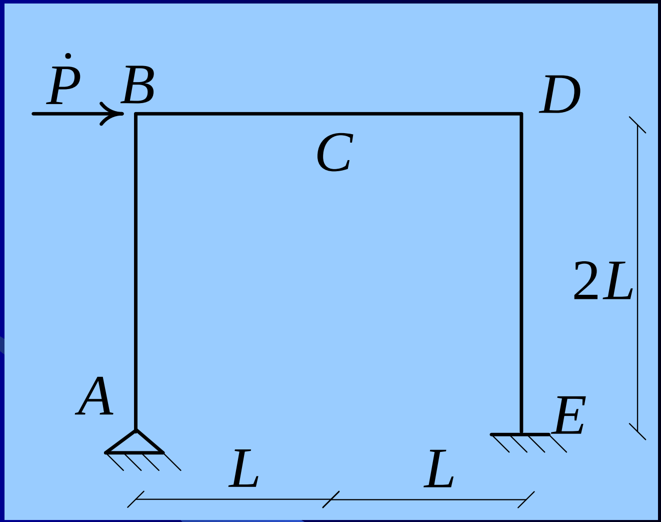
$$M_{ED} = -1.59493M_0$$

$$u_B = -754.20\times10^{-3}\frac{M_0L^2}{EI}$$

$$w_C = 420.46\times10^{-3}\frac{M_0L^2}{EI}$$



Step 8:



$$\dot{M}_{BA} = \dot{M}_{BC} = 0.43038\dot{P}L$$

$$\dot{M}_{CB} = \dot{M}_{CD} = -0.06329\dot{P}L$$

$$\dot{M}_{DC} = \dot{M}_{DE} = -0.55696\dot{P}L$$

$$\dot{M}_{ED} = 1.01266\dot{P}L$$

$$\dot{u}_B = \left(489.45\frac{\dot{P}L^3}{EI}\right)\times 10^{-3}$$

$$\dot{w}_C = \left(-31.65\frac{\dot{P}L^3}{EI}\right)\times 10^{-3}$$

Step 8, Loading O:

$$\dot{P} = 1.5\frac{M_0}{L}$$

$$M_{BA} = M_{BC} = 0.16772M_0$$

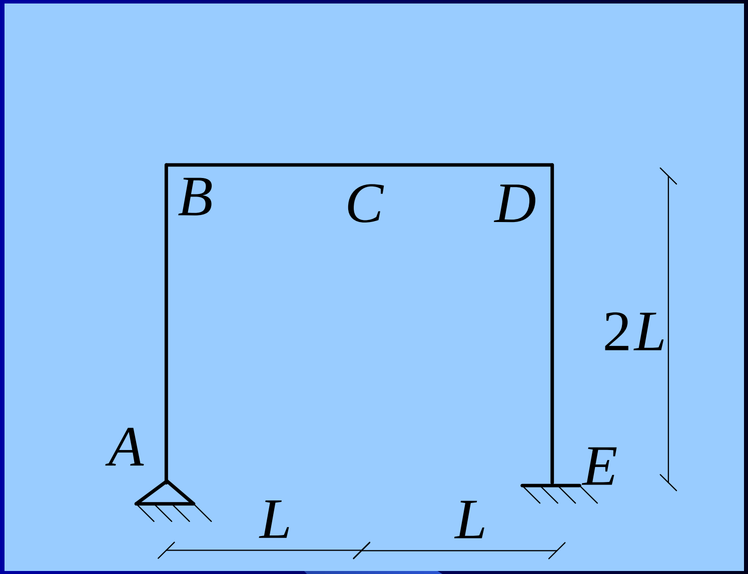
$$M_{CB} = M_{CD} = 0.12975M_0$$

$$M_{DC} = M_{DE} = 0.09129M_0$$

$$M_{ED} = -0.07594M_0$$

$$u_B = -20.03\times10^{-3}\frac{M_0L^2}{EI}$$

$$w_C = 372.99\times10^{-3}\frac{M_0L^2}{EI}$$



Step 8, Loading A:

$$\dot{P} = 1.5\frac{M_0}{L}$$

$$M_{BA} = M_{BC} = 0.81329 M_0$$

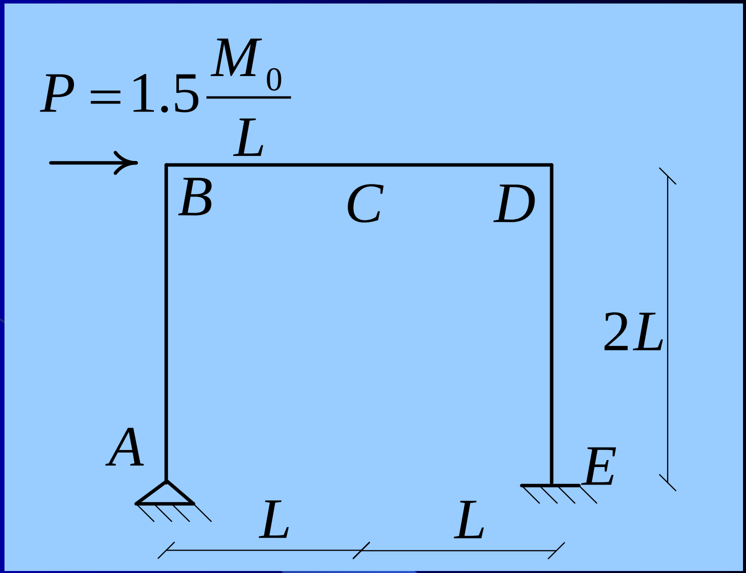
$$M_{CB} = M_{CD} = 0.03482 M_0$$

$$M_{DC} = M_{DE} = -0.74415 M_0$$

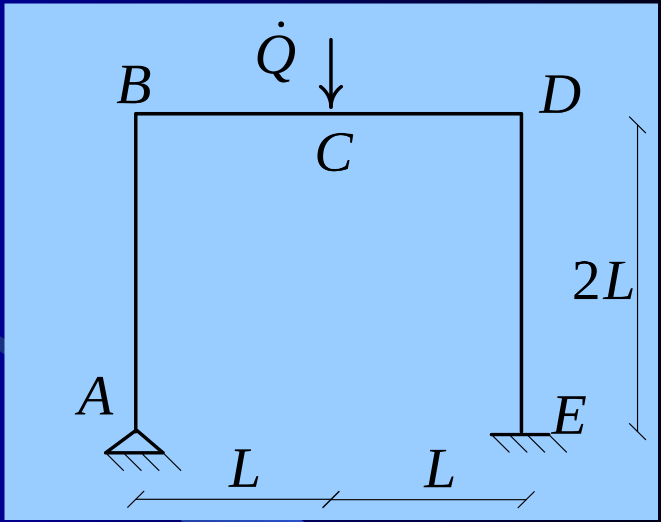
$$M_{ED} = 1.44305 M_0$$

$$u_B = 714.15\times10^{-3}\frac{M_0L^2}{EI}$$

$$w_C = 325.52\times10^{-3}\frac{M_0L^2}{EI}$$



Step 9:



$$\dot{M}_{BA} = \dot{M}_{BC} = -0.20886\dot{Q}L$$

$$\dot{M}_{CB} = \dot{M}_{CD} = 0.31013\dot{Q}L$$

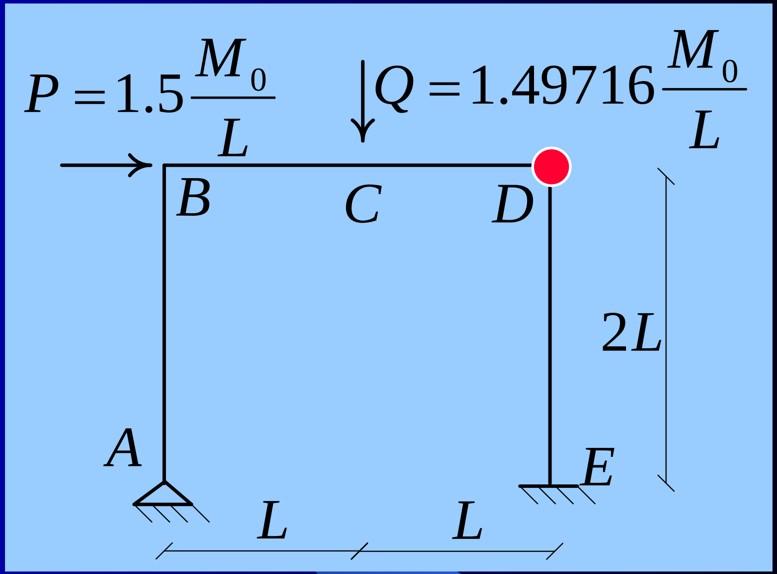
$$\dot{M}_{DC} = \dot{M}_{DE} = -0.17089\dot{Q}L$$

$$\dot{M}_{ED} = 0.03797\dot{Q}L$$

$$\dot{u}_B = \left(-31.65\frac{\dot{P}L^3}{EI} \right) \times 10^{-3}$$

$$\dot{w}_C = \left(71.73\frac{\dot{Q}L^3}{EI} \right) \times 10^{-3}$$

Step 9, Loading Between A and B:



$$\dot{Q} = 1.49716\frac{M_0}{L}$$

$$M_{BA} = M_{BC} = 0.50059 M_0$$

$$M_{CB} = M_{CD} = 0.49913 M_0$$

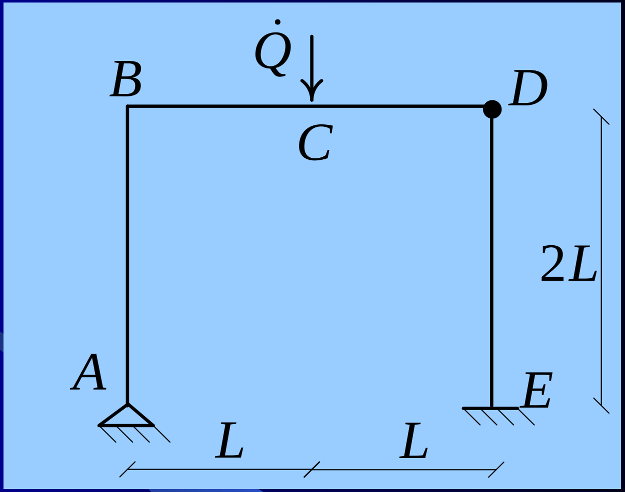
$$M_{DC} = M_{DE} = -M_0$$

$$M_{ED} = 1.49990 M_0$$

$$u_B = 666.76 \times 10^{-3}\frac{M_0 L^2}{EI}$$

$$w_C = 432.91 \times 10^{-3}\frac{M_0 L^2}{EI}$$

Step 10:



$$\dot{M}_{BA} = \dot{M}_{BC} = -0.18750\dot{Q}L$$

$$\dot{M}_{CB} = \dot{M}_{CD} = 0.40625\dot{Q}L$$

$$\dot{M}_{DC} = \dot{M}_{DE} = 0$$

$$\dot{M}_{ED} = 0.18750\dot{Q}L$$

$$\dot{u}_B = \left(125.00\frac{\dot{Q}L^3}{EI}\right)\times 10^{-3}$$

$$\dot{w}_C = \left(119.79\frac{\dot{Q}L^3}{EI}\right)\times 10^{-3}$$

Step 10, Loading B:

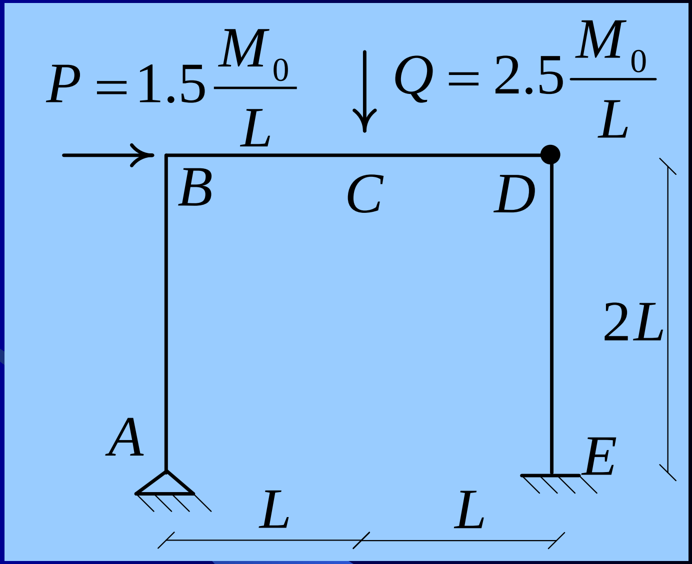
$$\dot{Q} = 1.00284\frac{M_0}{L}$$

$$M_{BA} = M_{BC} = 0.31256M_0$$

$$M_{CB} = M_{CD} = 0.90653M_0$$

$$M_{DC} = M_{DE} = -M_0$$

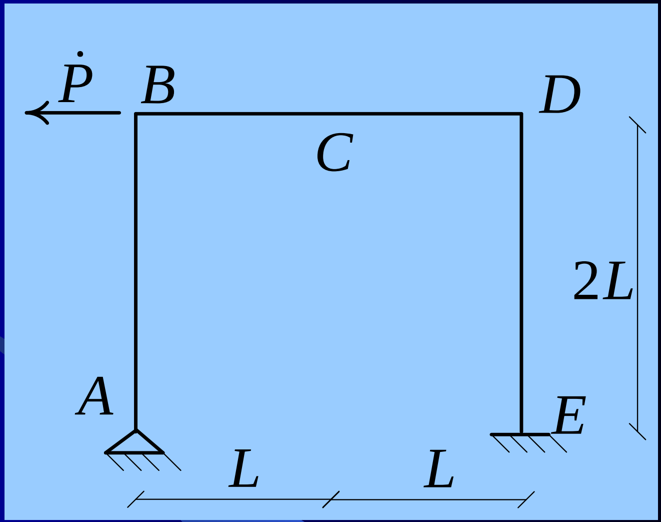
$$M_{ED} = 1.68793M_0$$



$$u_B = 792.12\times10^{-3}\frac{M_0L^2}{EI}$$

$$w_B = 553.04\times10^{-3}\frac{M_0L^2}{EI}$$

Step 11:



$$\dot{M}_{BA} = \dot{M}_{BC} = -0.43038\dot{P}L$$

$$\dot{M}_{CB} = \dot{M}_{CD} = 0.06329\dot{P}L$$

$$\dot{M}_{DC} = \dot{M}_{DE} = 0.55696\dot{P}L$$

$$\dot{M}_{ED} = -1.01266\dot{P}L$$

$$\dot{u}_B = \left(-489.45\frac{\dot{P}L^3}{EI}\right)\times 10^{-3}$$

$$\dot{w}_C = \left(31.65\frac{\dot{P}L^3}{EI}\right)\times 10^{-3}$$

Step 11, Loading between B and C:

$$\dot{P} = 1.47685\frac{M_0}{L}$$

$$M_{BA} = M_{BC} = -0.32305 M_0$$

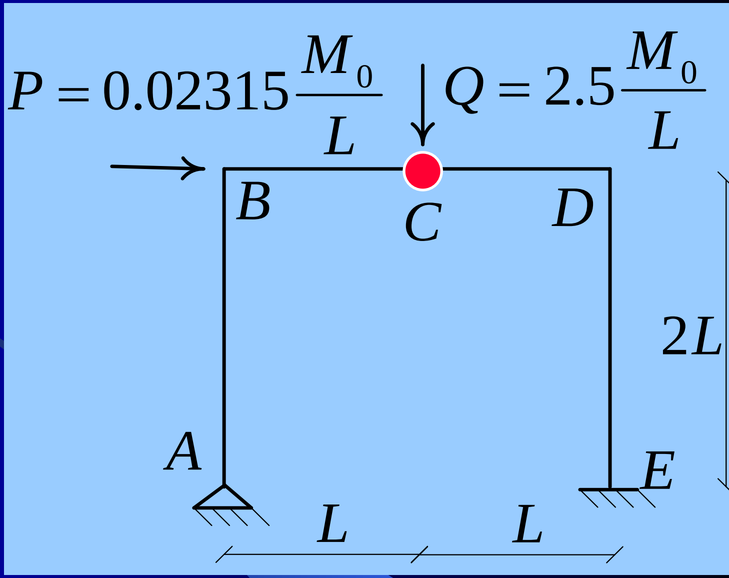
$$M_{CB} = M_{CD} = M_0$$

$$M_{DC} = M_{DE} = -0.17745 M_0$$

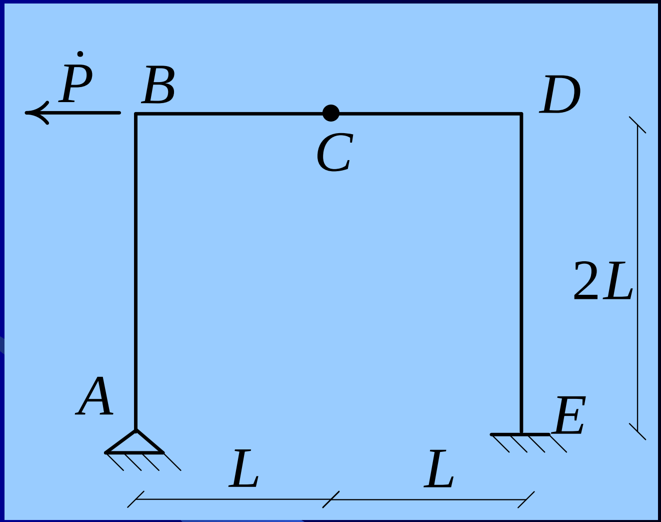
$$M_{ED} = 0.19238 M_0$$

$$u_B = 69.28\times10^{-3}\frac{M_0 L^2}{EI}$$

$$w_C = 579.78\times10^{-3}\frac{M_0 L^2}{EI}$$



Step 12:



$$\dot{M}_{BA} = \dot{M}_{BC} = -0.50000\dot{P}L$$

$$\dot{M}_{CB} = \dot{M}_{CD} = 0$$

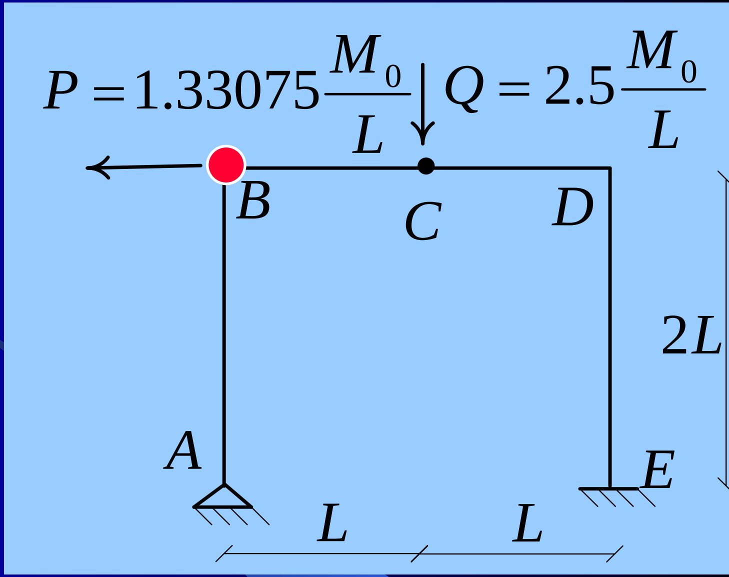
$$\dot{M}_{DC} = \dot{M}_{DE} = 0.50000\dot{P}L$$

$$\dot{M}_{ED} = -1.00000\dot{P}L$$

$$\dot{u}_B = \left(-500\frac{\dot{P}L^3}{EI} \right) \times 10^{-3}$$

$$\dot{w}_C = \left(83.33\frac{\dot{P}L^3}{EI} \right) \times 10^{-3}$$

Step 12, Loading between B and C:



$$\dot{P} = 1.35390\frac{M_0}{L}$$

$$M_{BA} = M_{BC} = -M_0$$

$$M_{CB} = M_{CD} = M_0$$

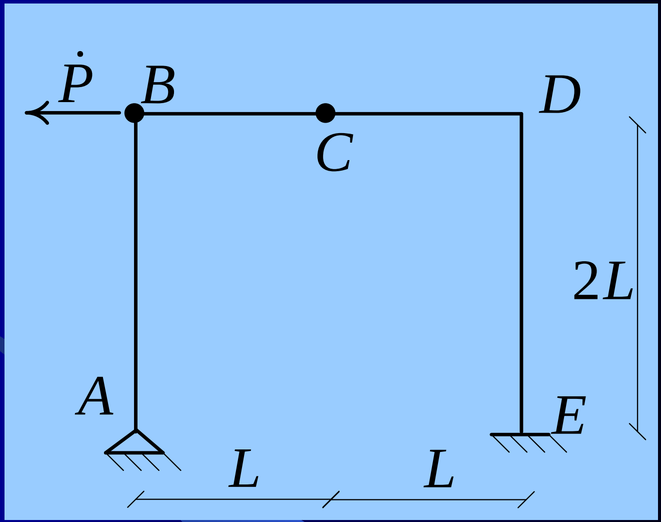
$$M_{DC} = M_{DE} = 0.49950M_0$$

$$M_{ED} = -1.16152M_0$$

$$u_B = -607.67\times10^{-3}\frac{M_0L^2}{EI}$$

$$w_C = 692.60\times10^{-3}\frac{M_0L^2}{EI}$$

Step 13:



$$\dot{M}_{BA} = \dot{M}_{BC} = 0$$

$$\dot{M}_{CB} = \dot{M}_{CD} = 0$$

$$\dot{M}_{DC} = \dot{M}_{DE} = 0$$

$$\dot{M}_{ED} = -2\dot{P}L$$

$$\dot{u}_B = \left(-1333.33\frac{\dot{Q}L^3}{EI}\right)\times 10^{-3}$$

$$\dot{w}_C = \left(1000.00\frac{\dot{Q}L^3}{EI}\right)\times 10^{-3}$$

Step 13, Loading C:

$$\dot{P} = 0.16925\frac{M_0}{L}$$

$$M_{BA} = M_{BC} = -M_0$$

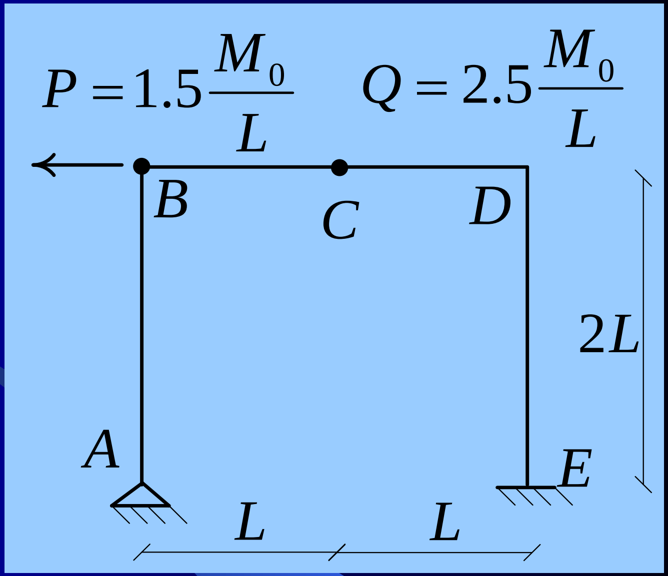
$$M_{CB} = M_{CD} = M_0$$

$$M_{DC} = M_{DE} = 0.49950M_0$$

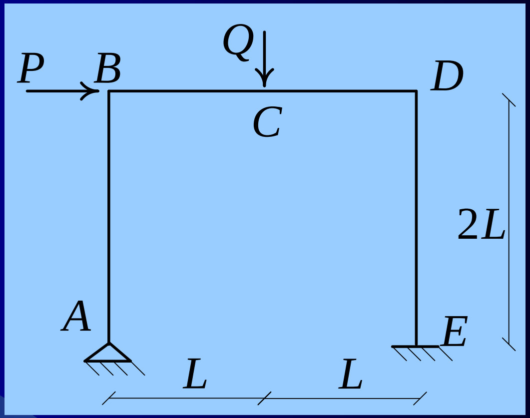
$$M_{ED} = -1.50002M_0$$

$$u_B = -833.33\times10^{-3}\frac{M_0L^2}{EI}$$

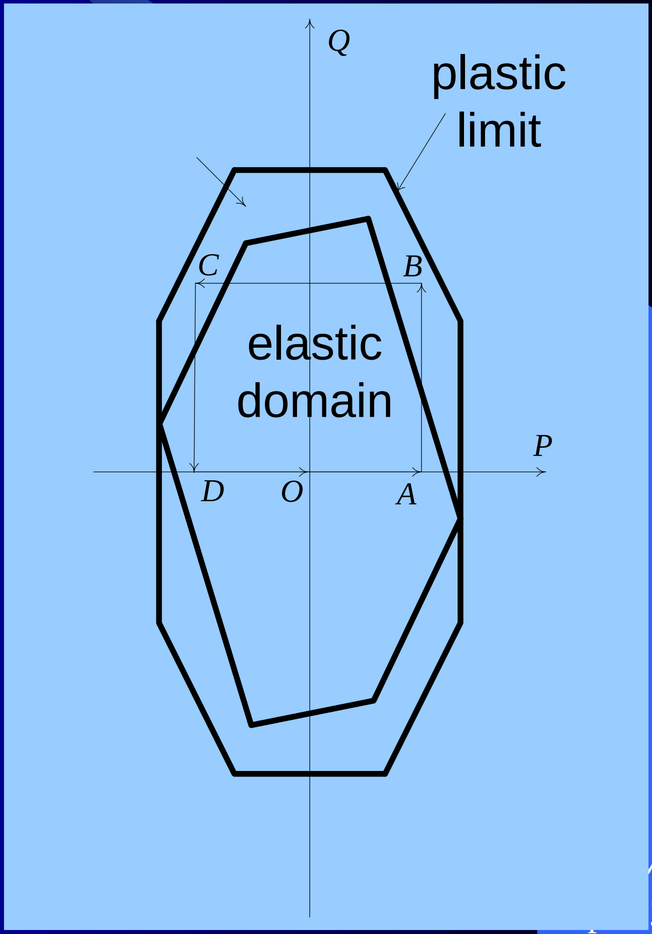
$$w_C = 861.85\times10^{-3}\frac{M_0L^2}{EI}$$



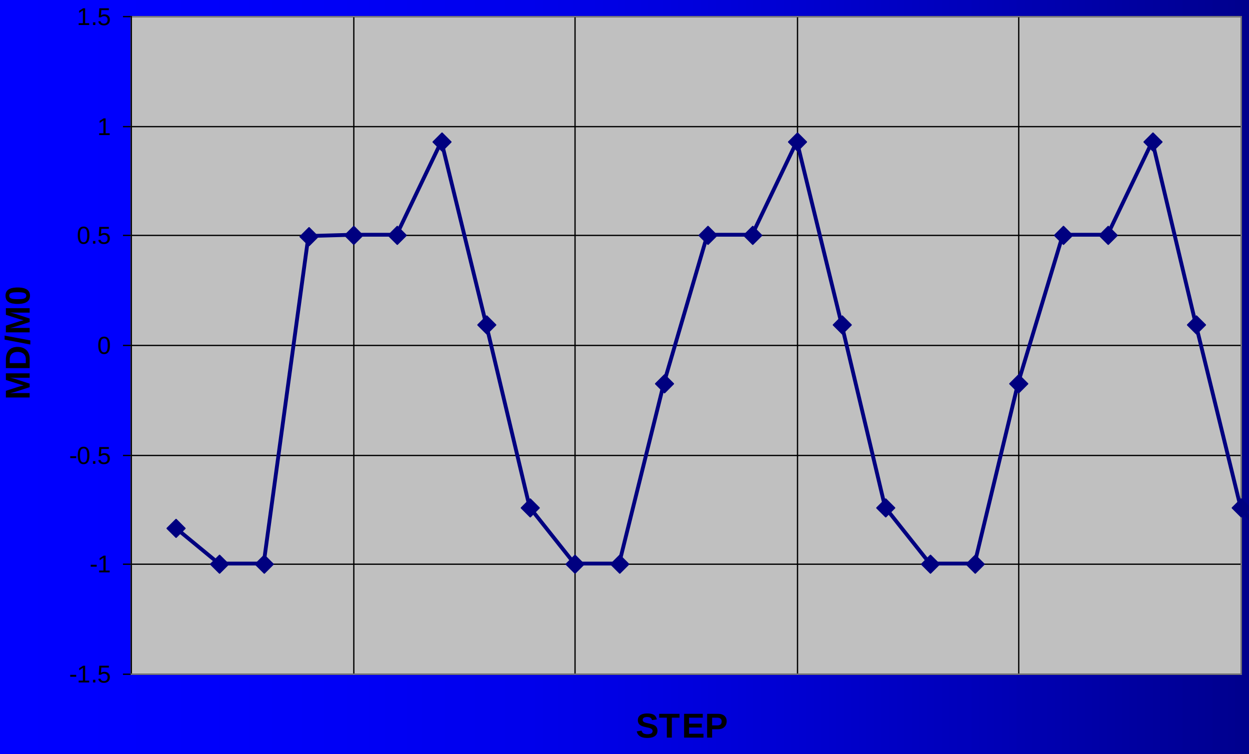
-Cyclic Loading, example, continue…:

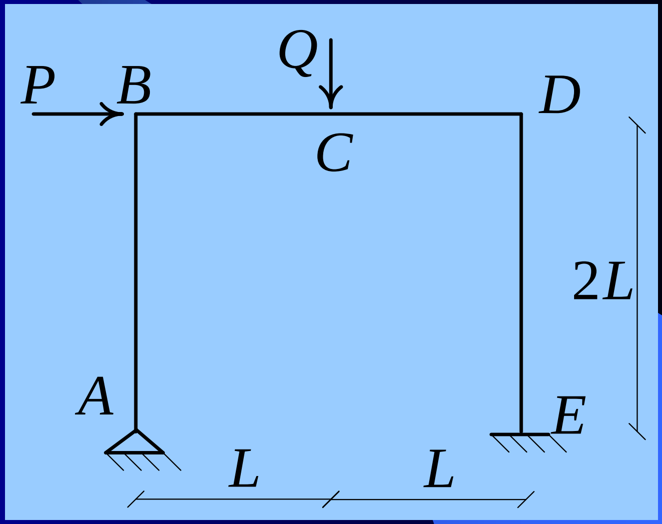


	M_B/M_0	M_C/M_0	M_D/M_0	M_E/M_0	$u_B/(M_0L^2/2EI)$	$w_C/(M_0L^2/2EI)$
A	0.64557	-0.09494	-0.83544	1.51899	734.18×10^{-3}	-47.48×10^{-3}
A-B	0.44506	0.20278	-1	1.55544	703.80×10^{-3}	21.38×10^{-3}
B	0.15631	0.82841	-1	1.84419	896.30×10^{-3}	205.86×10^{-3}
B-C	-1	0.99845	0.49640	-0.87654	-418.72×10^{-3}	290.89×10^{-3}
B-C	-1	1	0.49950	-0.88378	-422.51×10^{-3}	291.67×10^{-3}
C	-1	1	0.49950	-1.50000	-833.32×10^{-3}	599.78×10^{-3}
D	-0.47785	0.22468	0.92673	-1.59493	-754.20×10^{-3}	420.46×10^{-3}
O	0.16772	0.12975	0.09129	-0.07594	-20.03×10^{-3}	372.99×10^{-3}
A	0.81329	0.03482	-0.74415	1.44305	714.15×10^{-3}	325.52×10^{-3}
A-B	0.50059	0.49913	-1	1.49990	666.76×10^{-3}	432.91×10^{-3}
B	0.31256	0.90653	-1	1.68793	792.12×10^{-3}	533.04×10^{-3}
B-C	-0.32305	1	-0.17745	0.19238	69.28×10^{-3}	579.78×10^{-3}
B-C	-1	1	0.49950	-1.16152	-607.67×10^{-3}	692.60×10^{-3}
C	-1	1	0.49950	-1.50002	-833.33×10^{-3}	861.85×10^{-3}

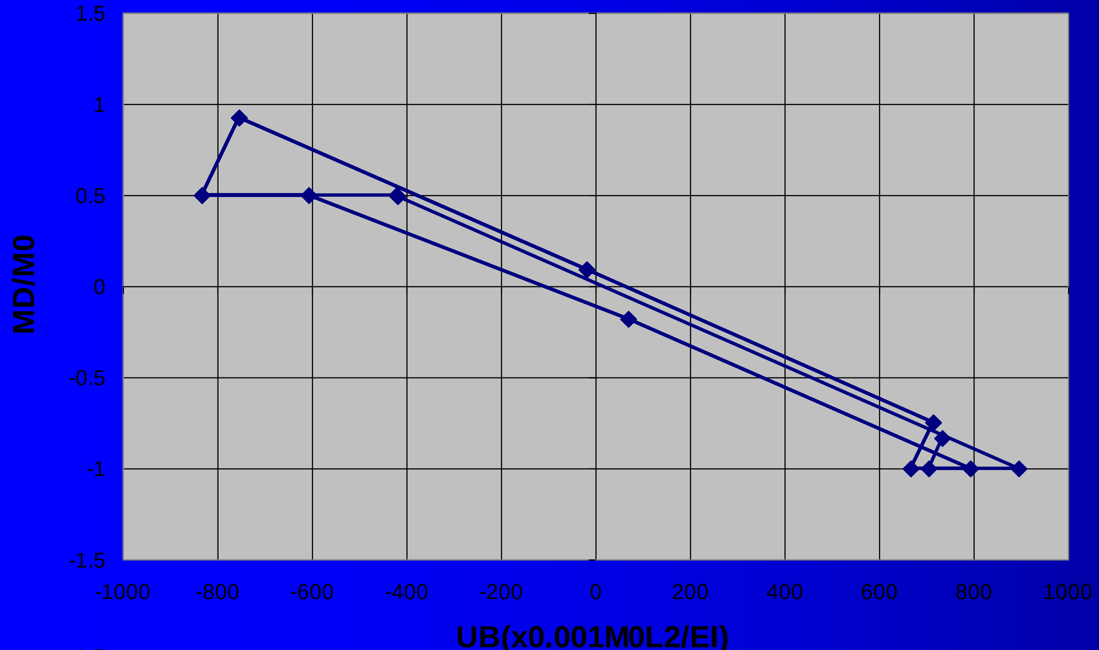


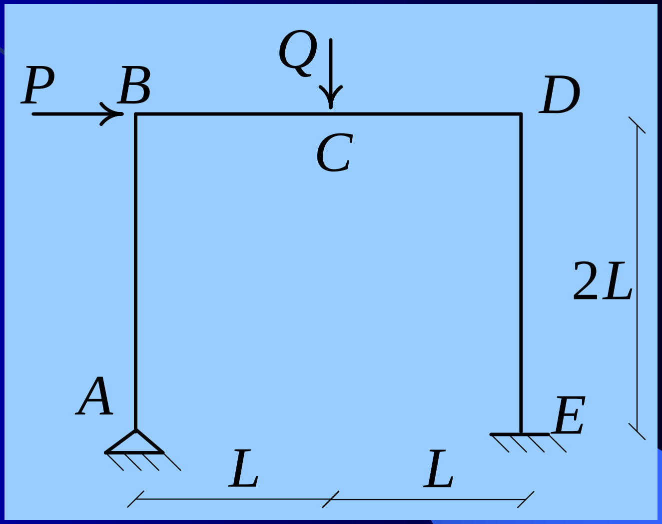
-Cyclic Loading, example, continue…:

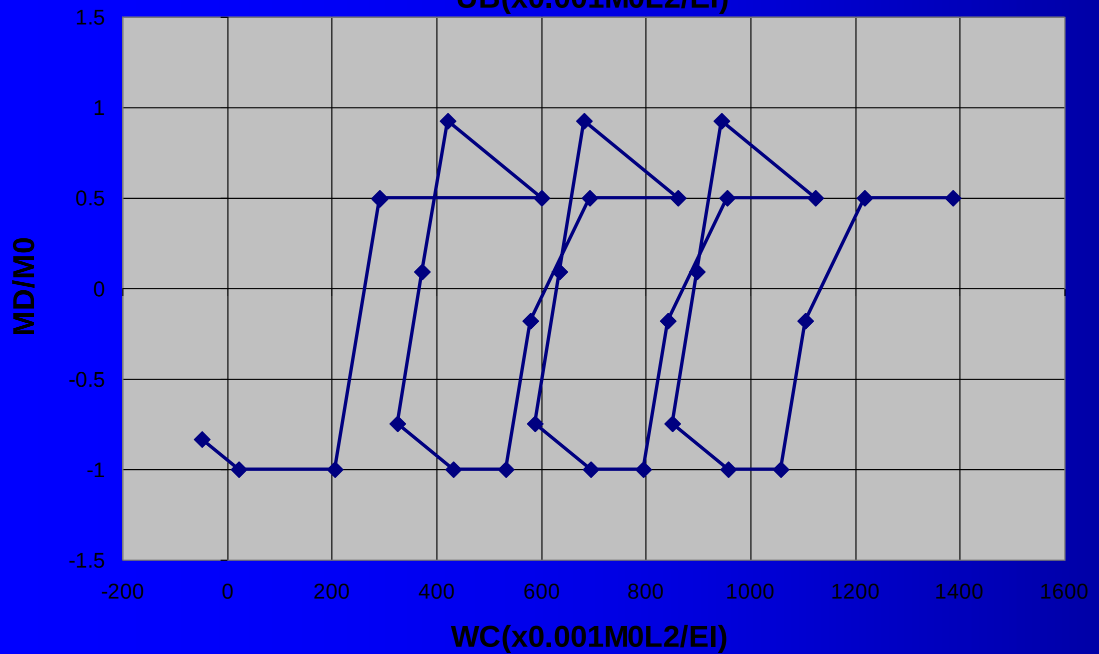




-Cyclic Loading, example, continue…:







Incremental Collapse

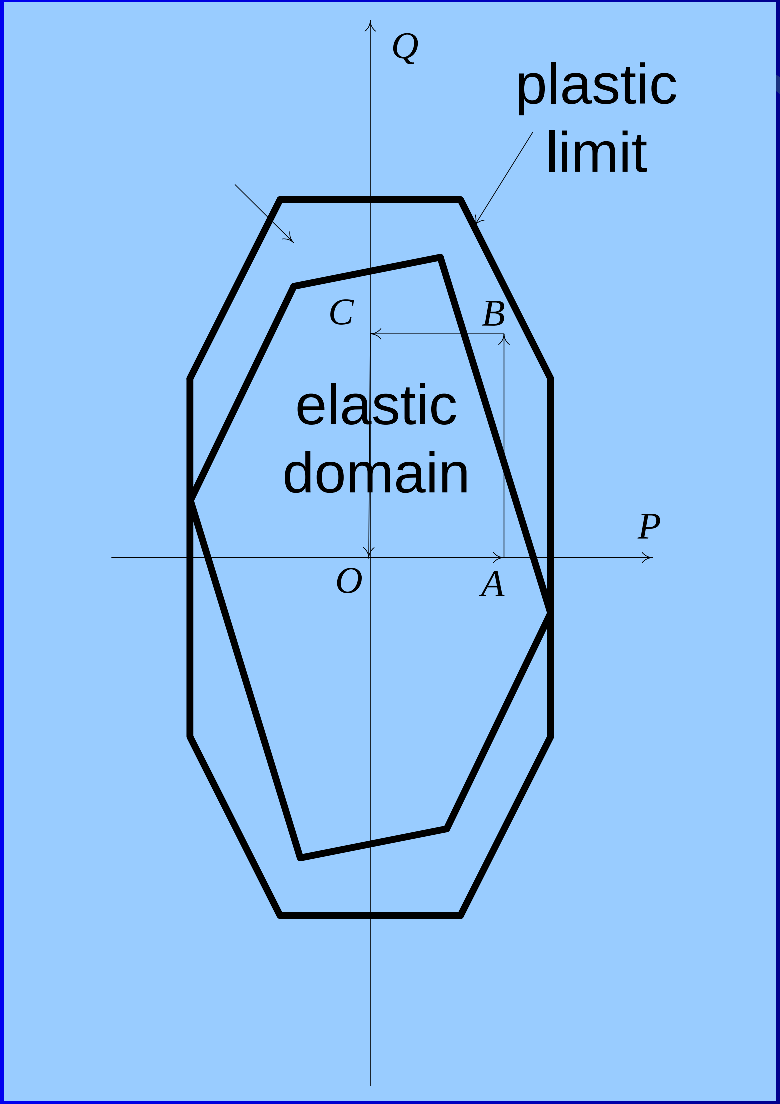
-Cyclic Loading, example:

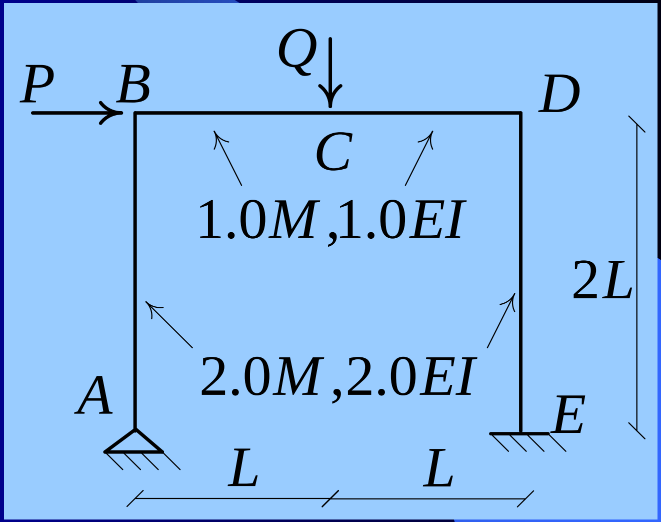
$$O:\begin{cases} P=0 \\ Q=0 \end{cases}$$

$$A:\begin{cases} P=\dfrac{1.5M_0}{L} \\ Q=0 \end{cases}$$

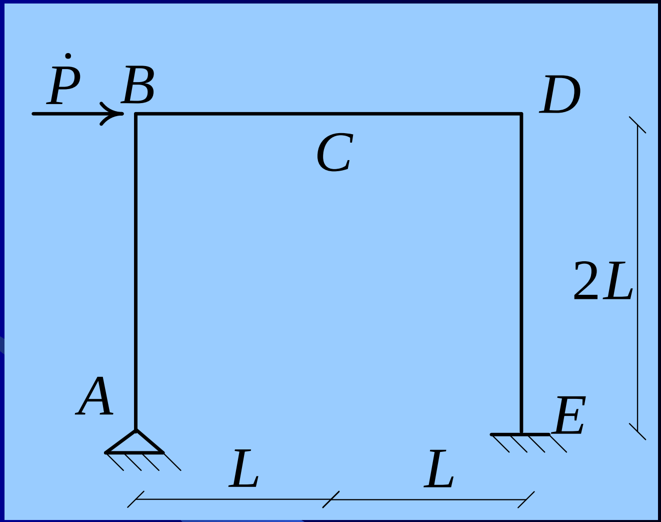
$$B:\begin{cases} P=\dfrac{1.5M_0}{L} \\ Q=\dfrac{2.5M_0}{L} \end{cases}$$

$$C:\begin{cases} P=0 \\ Q=\dfrac{2.5M_0}{L} \end{cases}$$





Step 1:



$$\dot{M}_{BA} = \dot{M}_{BC} = 0.43038\dot{P}L$$

$$\dot{M}_{CB} = \dot{M}_{CD} = -0.06329\dot{P}L$$

$$\dot{M}_{DC} = \dot{M}_{DE} = -0.55696\dot{P}L$$

$$\dot{M}_{ED} = 1.01266\dot{P}L$$

$$\dot{u}_B = \left(489.45\frac{\dot{P}L^3}{EI}\right)\times 10^{-3}$$

$$\dot{w}_C = \left(-31.65\frac{\dot{P}L^3}{EI}\right)\times 10^{-3}$$

Step 1, Loading A:

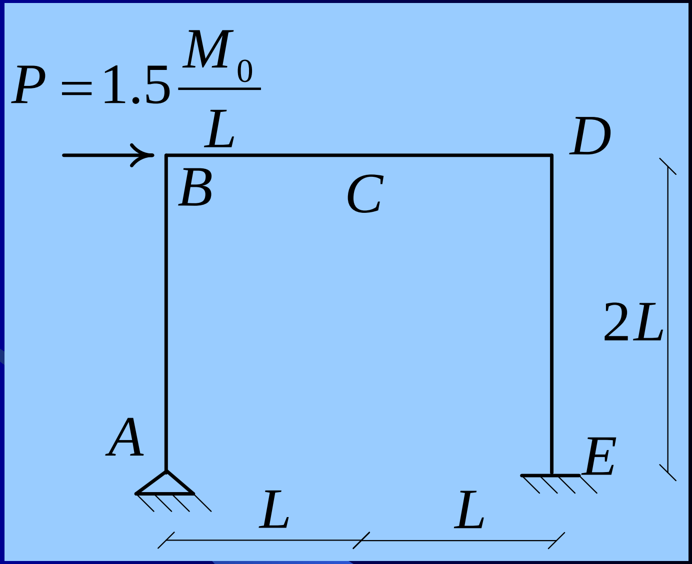
$$\dot{P} = 1.5\frac{M_0}{L}$$

$$M_{BA} = M_{BC} = 0.64557M_0$$

$$M_{CB} = M_{CD} = -0.09494M_0$$

$$M_{DC} = M_{DE} = -0.83544M_0$$

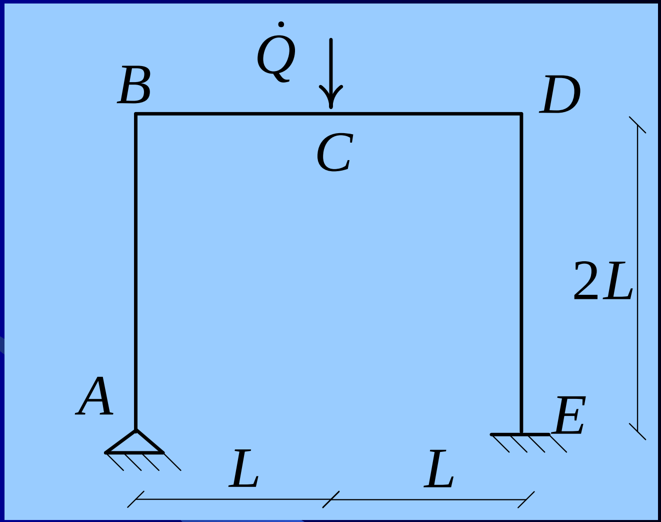
$$M_{ED} = 1.51899M_0$$



$$u_B = 734.18\times10^{-3}\frac{M_0L^2}{EI}$$

$$w_C = -47.48\times10^{-3}\frac{M_0L^2}{EI}$$

Step 2:



$$\dot{M}_{BA} = \dot{M}_{BC} = -0.20886\dot{Q}L$$

$$\dot{M}_{CB} = \dot{M}_{CD} = 0.31013\dot{Q}L$$

$$\dot{M}_{DC} = \dot{M}_{DE} = -0.17089\dot{Q}L$$

$$\dot{M}_{ED} = 0.03797\dot{Q}L$$

$$\dot{u}_B = \left(-31.65\frac{\dot{P}L^3}{EI}\right)\times 10^{-3}$$

$$\dot{w}_C = \left(71.73\frac{\dot{Q}L^3}{EI}\right)\times 10^{-3}$$

Step 2, Loading between A and B:

$$\dot{Q} = 0.96\frac{M_0}{L}$$

$$M_{BA} = M_{BC} = 0.44506M_0$$

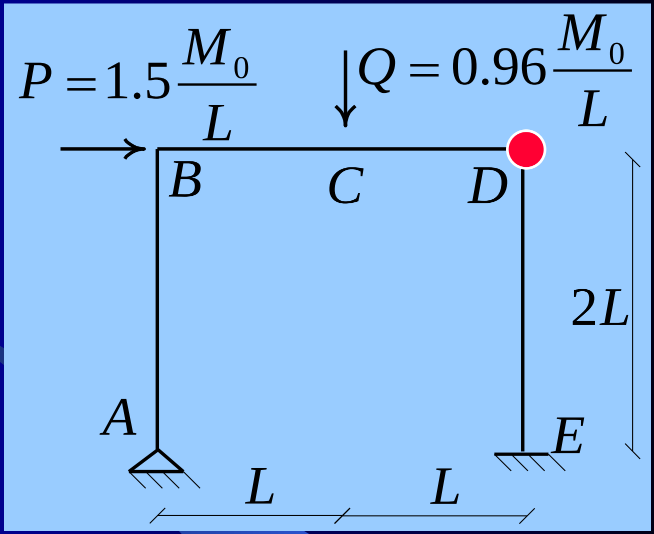
$$M_{CB} = M_{CD} = 0.20278M_0$$

$$M_{DC} = M_{DE} = -M_0$$

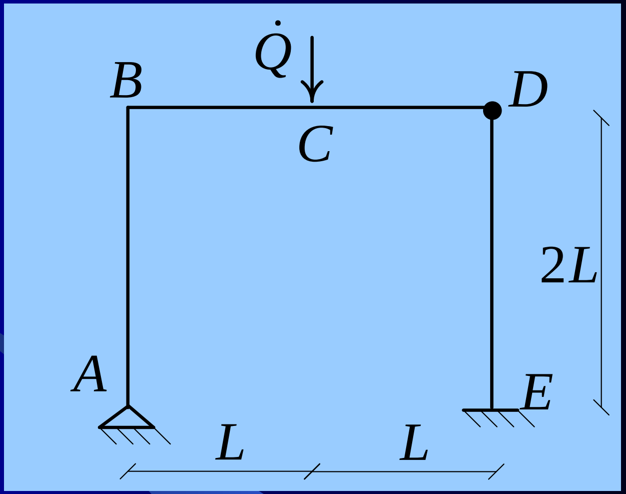
$$M_{ED} = 1.55544M_0$$

$$u_B = 703.80\times10^{-3}\frac{M_0L^2}{EI}$$

$$w_C = 21.38\times10^{-3}\frac{M_0L^2}{EI}$$



Step 3:



$$\dot{M}_{BA} = \dot{M}_{BC} = -0.18750\dot{Q}L$$

$$\dot{M}_{CB} = \dot{M}_{CD} = 0.40625\dot{Q}L$$

$$\dot{M}_{DC} = \dot{M}_{DE} = 0$$

$$\dot{M}_{ED} = 0.18750\dot{Q}L$$

$$\dot{u}_B = \left(125.00\frac{\dot{Q}L^3}{EI}\right)\times 10^{-3}$$

$$\dot{w}_C = \left(119.79\frac{\dot{Q}L^3}{EI}\right)\times 10^{-3}$$

Step 3, Loading B:

$$\dot{Q} = 1.54\frac{M_0}{L}$$

$$M_{BA} = M_{BC} = 0.15631M_0$$

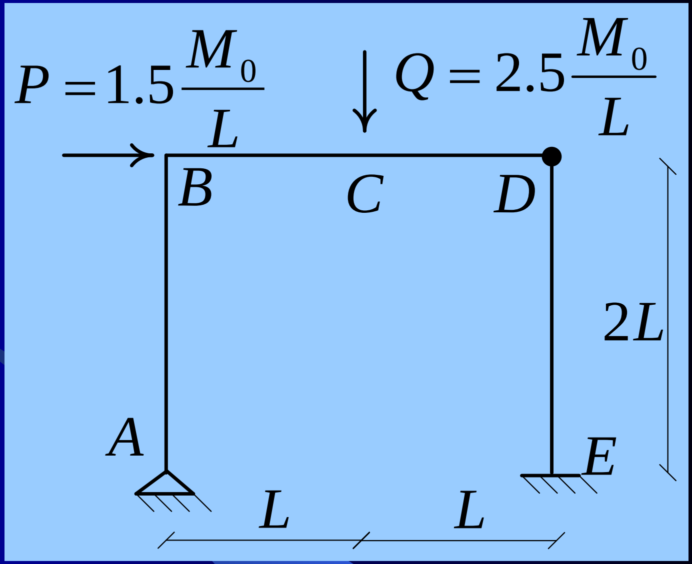
$$M_{CB} = M_{CD} = 0.82841M_0$$

$$M_{DC} = M_{DE} = -M_0$$

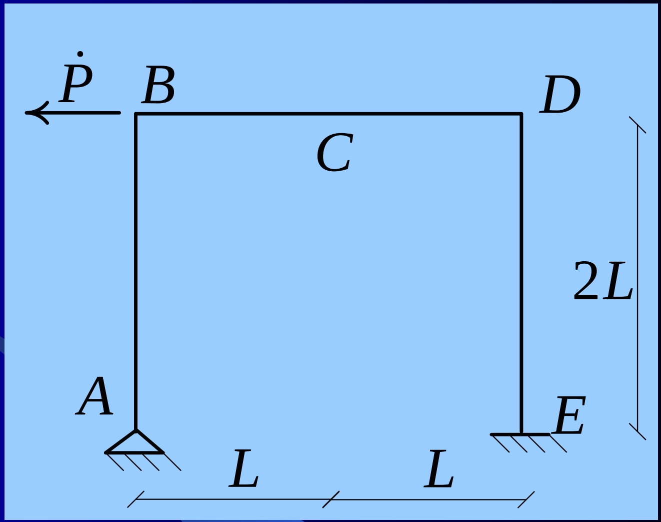
$$M_{ED} = 1.84419M_0$$

$$u_B = 896.30\times10^{-3}\frac{M_0L^2}{EI}$$

$$w_C = 205.86\times10^{-3}\frac{M_0L^2}{EI}$$



Step 4:



$$\dot{M}_{BA} = \dot{M}_{BC} = -0.43038\dot{P}L$$

$$\dot{M}_{CB} = \dot{M}_{CD} = 0.06329\dot{P}L$$

$$\dot{M}_{DC} = \dot{M}_{DE} = 0.55696\dot{P}L$$

$$\dot{M}_{ED} = -1.01266\dot{P}L$$

$$\dot{u}_B = \left(-489.45\frac{\dot{P}L^3}{EI}\right)\times 10^{-3}$$

$$\dot{w}_C = \left(31.65\frac{\dot{P}L^3}{EI}\right)\times 10^{-3}$$

Step 4, Loading C:

$$\dot{P} = 1.5\frac{M_0}{L}$$

$$M_{BA} = M_{BC} = -0.48926 M_0$$

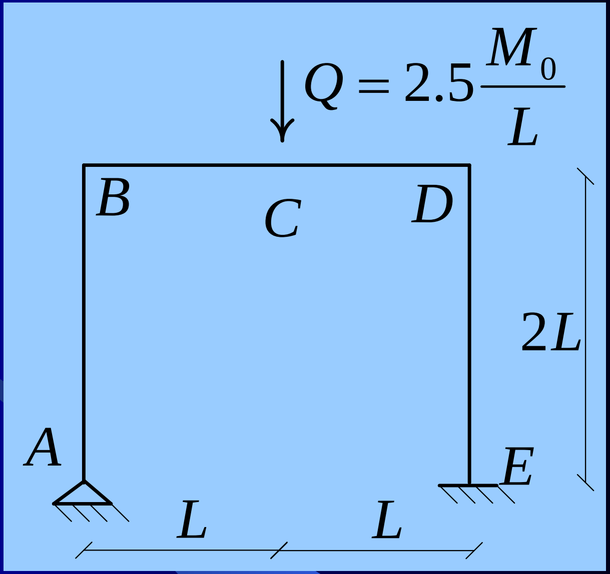
$$M_{CB} = M_{CD} = 0.92335 M_0$$

$$M_{DC} = M_{DE} = -0.16456 M_0$$

$$M_{ED} = 0.32520 M_0$$

$$u_B = 162.08\times10^{-3}\frac{M_0 L^2}{EI}$$

$$w_C = 253.34\times10^{-3}\frac{M_0 L^2}{EI}$$



Step 5:

$$\dot{M}_{BA} = \dot{M}_{BC} = 0.20886\dot{Q}L$$

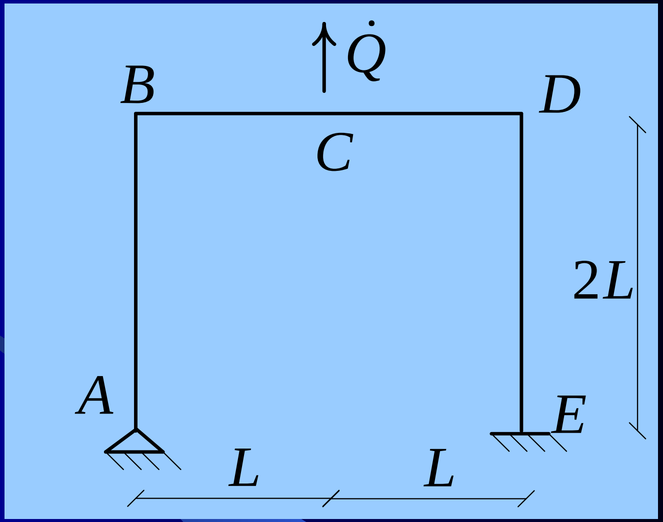
$$\dot{M}_{CB} = \dot{M}_{CD} = -0.31013\dot{Q}L$$

$$\dot{M}_{DC} = \dot{M}_{DE} = 0.17089\dot{Q}L$$

$$\dot{M}_{ED} = -0.03797\dot{Q}L$$

$$\dot{u}_B = \left(31.65\frac{\dot{P}L^3}{EI}\right)\times 10^{-3}$$

$$\dot{w}_C = \left(-71.73\frac{\dot{Q}L^3}{EI}\right)\times 10^{-3}$$



Step 5, Loading O:

$$\dot{Q} = 2.5\frac{M_0}{L}$$

$$M_{BA} = M_{BC} = 0.03289 M_0$$

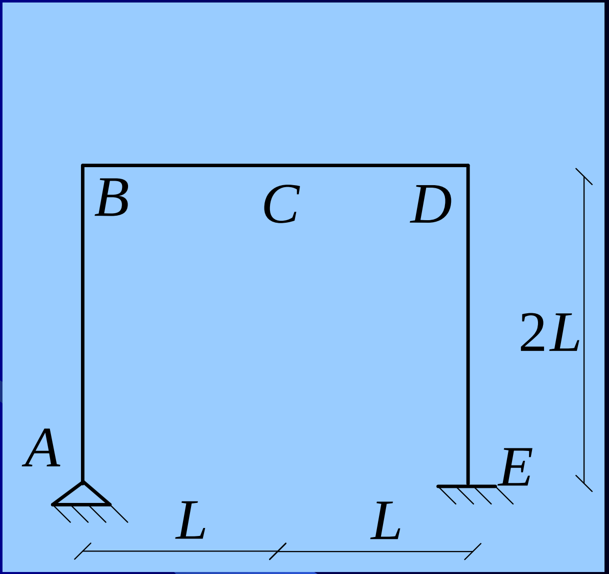
$$M_{CB} = M_{CD} = 0.14803 M_0$$

$$M_{DC} = M_{DE} = 0.26267 M_0$$

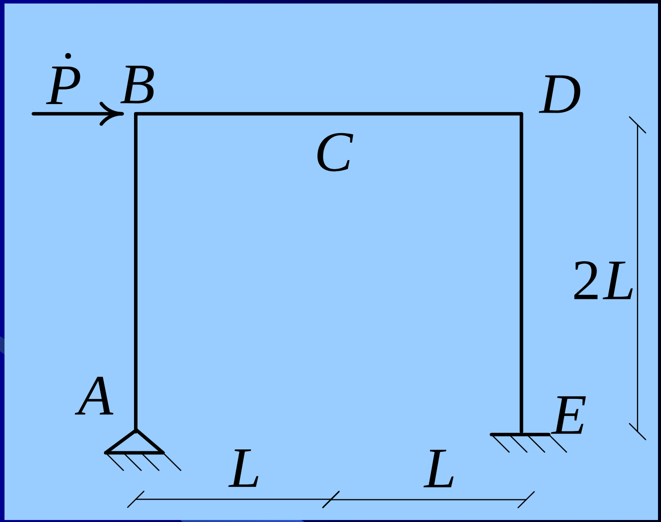
$$M_{ED} = 0.23028 M_0$$

$$u_B = 241.21\times10^{-3}\frac{M_0L^2}{EI}$$

$$w_C = 74.02\times10^{-3}\frac{M_0L^2}{EI}$$



Step 6:



$$\dot{M}_{BA} = \dot{M}_{BC} = 0.43038\dot{P}L$$

$$\dot{M}_{CB} = \dot{M}_{CD} = -0.06329\dot{P}L$$

$$\dot{M}_{DC} = \dot{M}_{DE} = -0.55696\dot{P}L$$

$$\dot{M}_{ED} = 1.01266\dot{P}L$$

$$\dot{u}_B = \left(489.45\frac{\dot{P}L^3}{EI}\right)\times 10^{-3}$$

$$\dot{w}_C = \left(-31.65\frac{\dot{P}L^3}{EI}\right)\times 10^{-3}$$

Step 6, Loading A:

$$\dot{P} = 1.5\frac{M_0}{L}$$

$$M_{BA} = M_{BC} = 0.67846 M_0$$

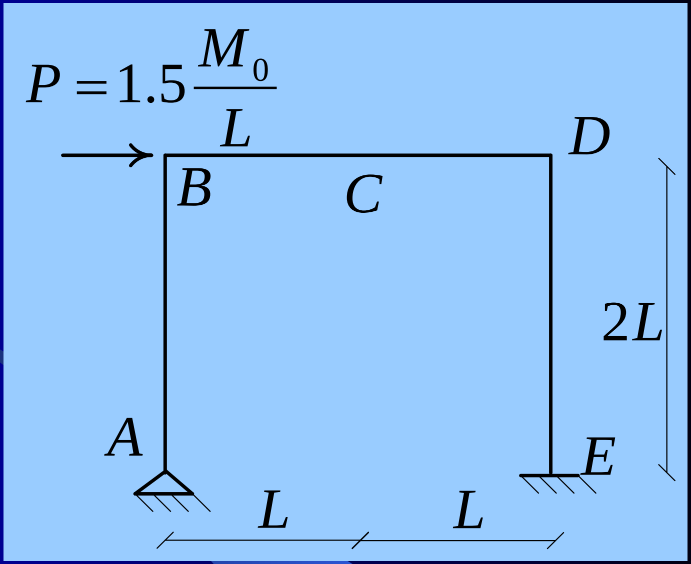
$$M_{CB} = M_{CD} = 0.05310 M_0$$

$$M_{DC} = M_{DE} = -0.57277 M_0$$

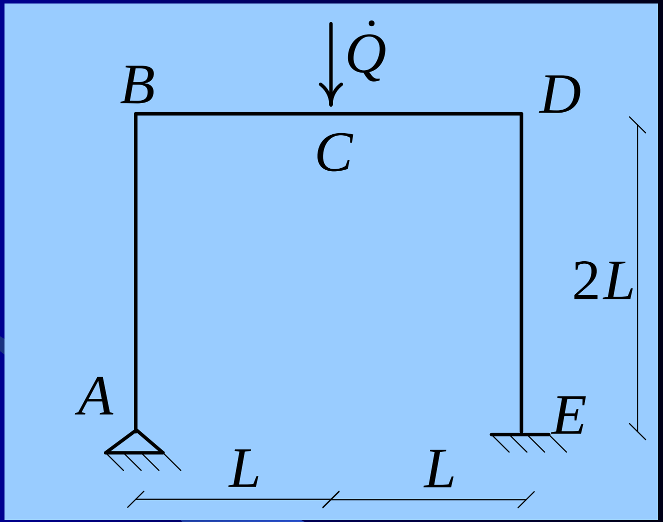
$$M_{ED} = 1.74927 M_0$$

$$u_B = 975.38\times 10^{-3}\frac{M_0 L^2}{EI}$$

$$w_C = 26.55\times 10^{-3}\frac{M_0 L^2}{EI}$$



Step 7:



$$\dot{M}_{BA} = \dot{M}_{BC} = -0.20886\dot{Q}L$$

$$\dot{M}_{CB} = \dot{M}_{CD} = 0.31013\dot{Q}L$$

$$\dot{M}_{DC} = \dot{M}_{DE} = -0.17089\dot{Q}L$$

$$\dot{M}_{ED} = 0.03797\dot{Q}L$$

$$\dot{u}_B = \left(-31.65\frac{\dot{P}L^3}{EI} \right) \times 10^{-3}$$

$$\dot{w}_C = \left(71.73\frac{\dot{Q}L^3}{EI} \right) \times 10^{-3}$$

Step 7, Loading B:

$$\dot{Q} = 2.5\frac{M_0}{L}$$

$$M_{BA} = M_{BC} = 0.15631 M_0$$

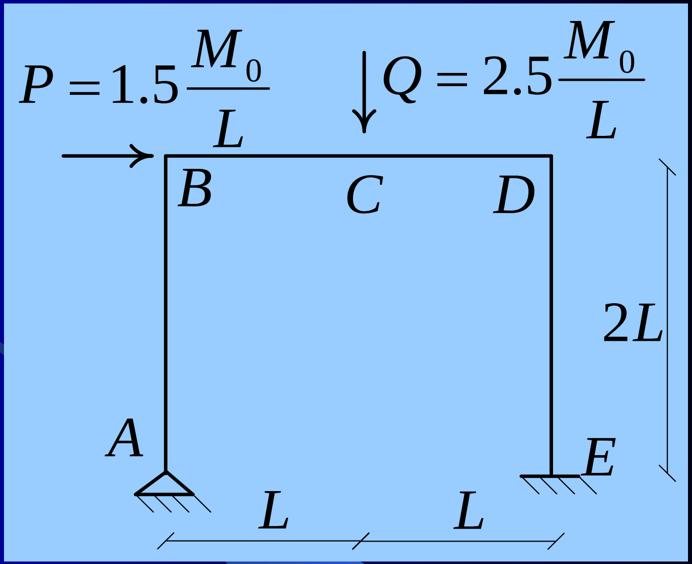
$$M_{CB} = M_{CD} = 0.82843 M_0$$

$$M_{DC} = M_{DE} = -M_0$$

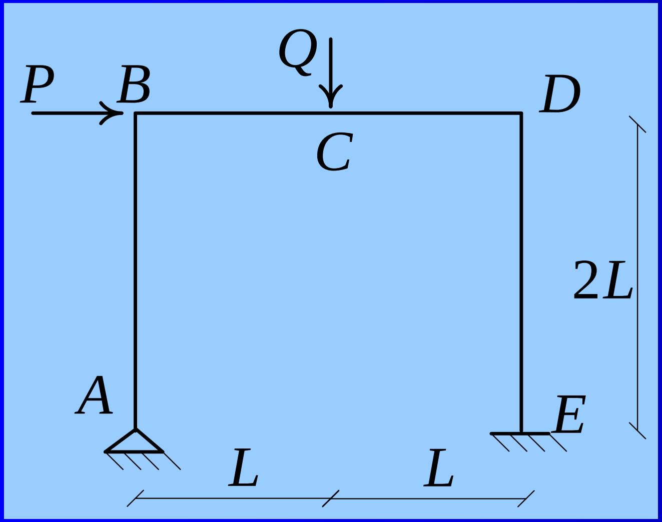
$$M_{ED} = 1.84419 M_0$$

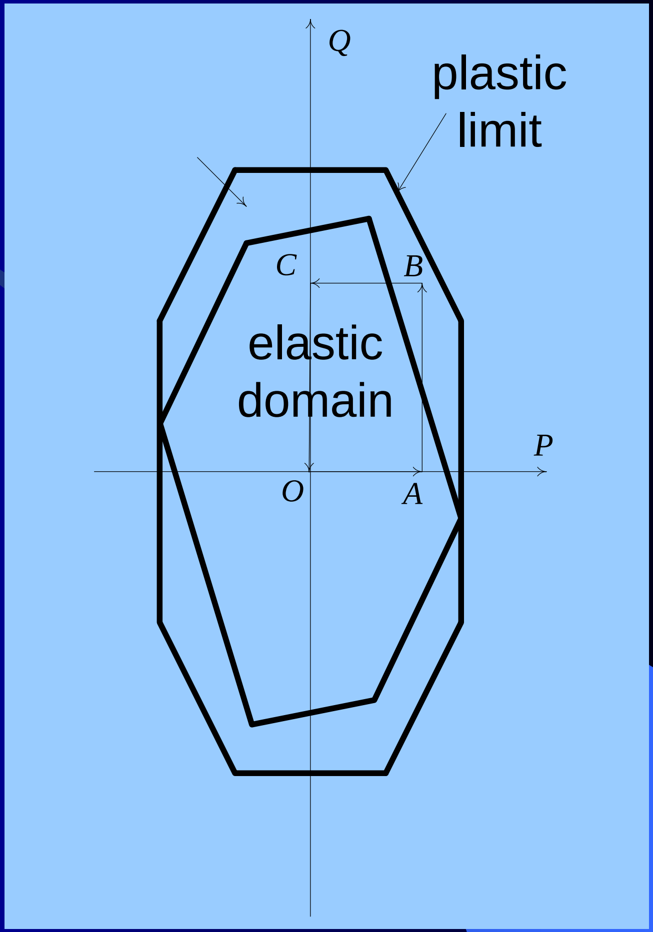
$$u_B = 896.26\times10^{-3}\frac{M_0 L^2}{EI}$$

$$w_C = 205.88\times10^{-3}\frac{M_0 L^2}{EI}$$



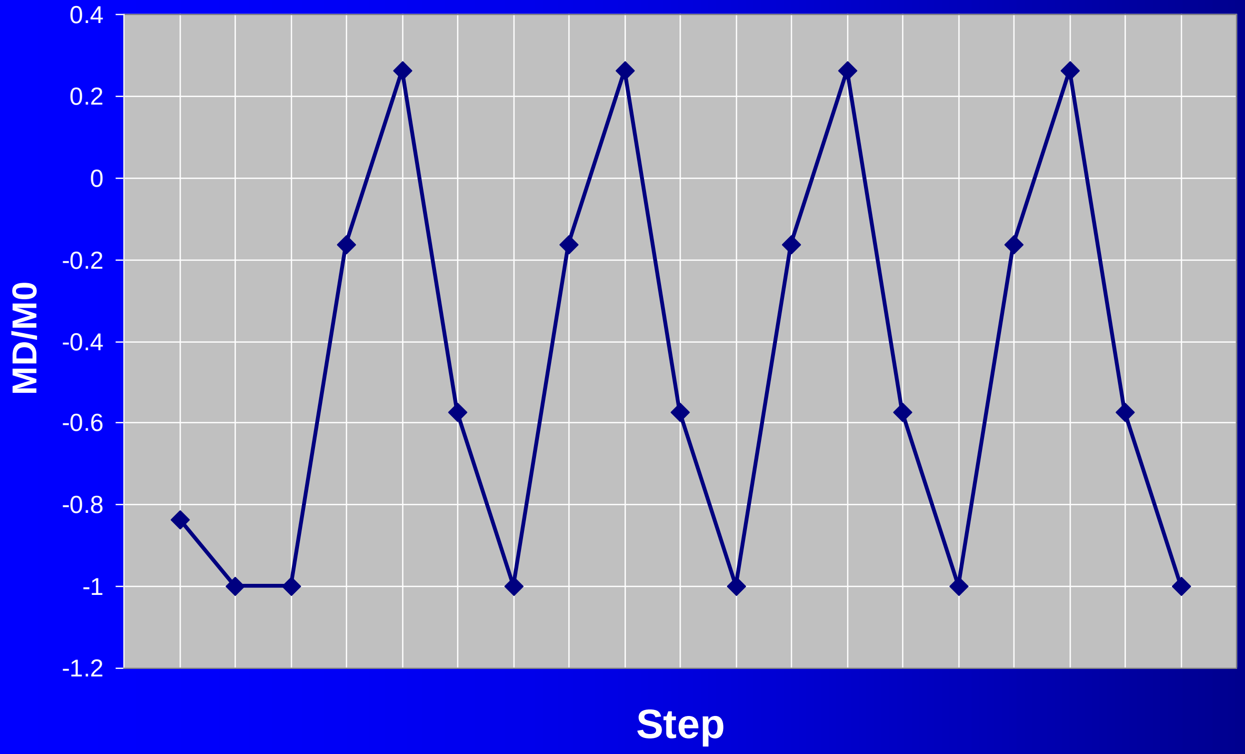
-Cyclic Loading, example, continue…:

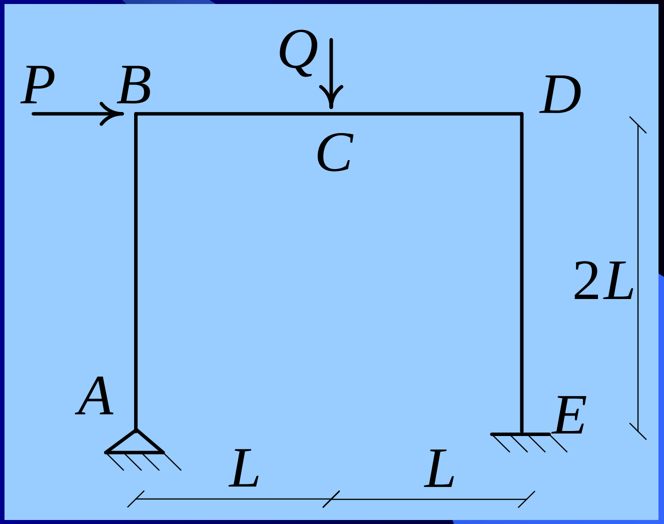




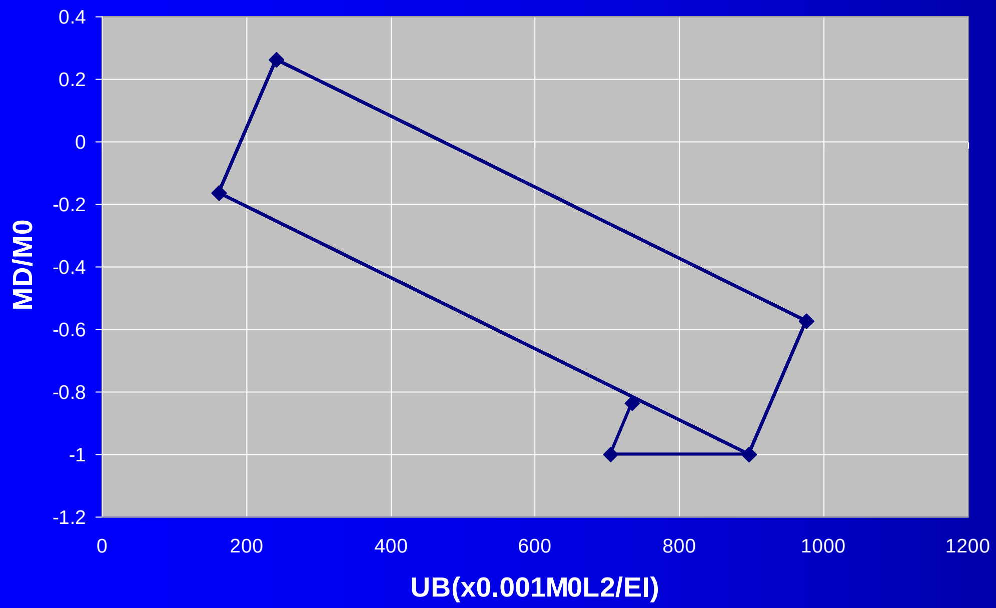
	M_B/M_0	M_C/M_0	M_D/M_0	M_E/M_0	$u_B/(M_0L^2/2EI)$	$w_C/(M_0L^2/2EI)$
A	0.64557	-0.09494	-0.83544	1.51899	734.18x10-3	-47.48x10-3
A-B	0.44506	0.20278	-1	1.55544	703.80x10-3	21.38x10-3
B	0.15631	0.82841	-1	1.84419	896.30x10-3	205.86x10-3
C	-0.48926	0.92335	-0.16456	0.32520	162.08 x10-3	253.34x10-3
O	0.03289	0.14803	0.26267	0.23028	241.21x10-3	74.02x10-3
A	0.67846	0.05310	-0.57277	1.74927	975.38x10-3	26.55x10-3
B	0.15631	0.82843	-1	1.84419	896.26x10-3	205.88x10-3

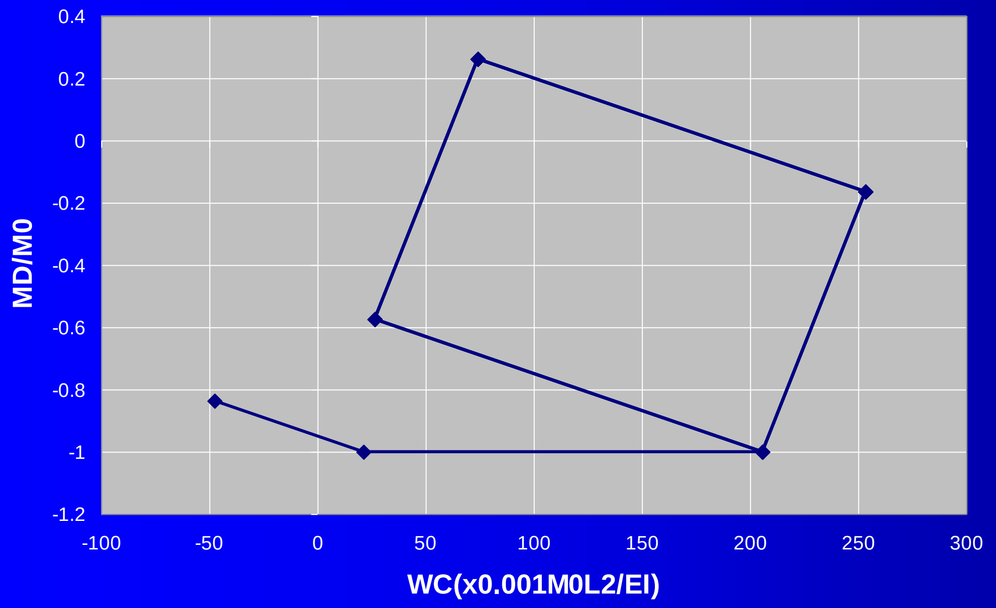
-Cyclic Loading, example, continue…:

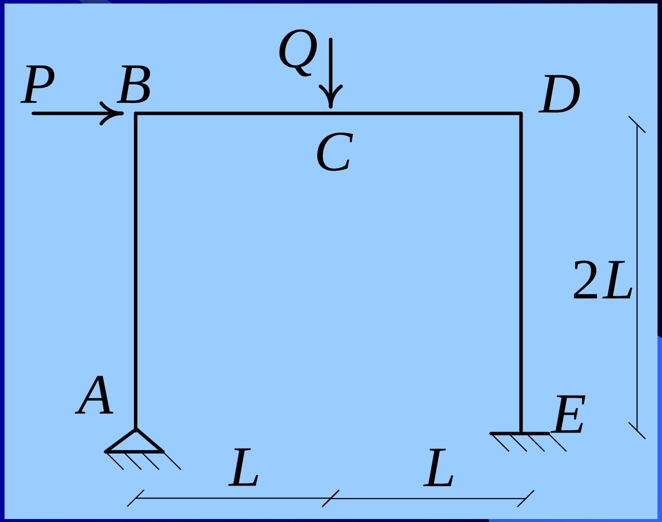




-Cyclic Loading, example, continue…:







Shake Down

-Melan's shakedown theorem:

-If for a given load history, there exist a self equilibrated shakedown forces $\mathbf{s}_r$ such that:

$$-\mathbf{s}_0 < \mathbf{s}_r + \mathbf{s}_e(t) < \mathbf{s}_0$$

Then the structure shakes down.

Where $\mathbf{s}_e(t)$ is the elastic internal loads.

- In a loading domain the following conditions are used for a shakedown analysis:

$$\mathbf{s}_r + \mathbf{s}_e^{max} < \mathbf{s}_0$$

$$\mathbf{s}_r + \mathbf{s}_e^{min} > -\mathbf{s}_0$$

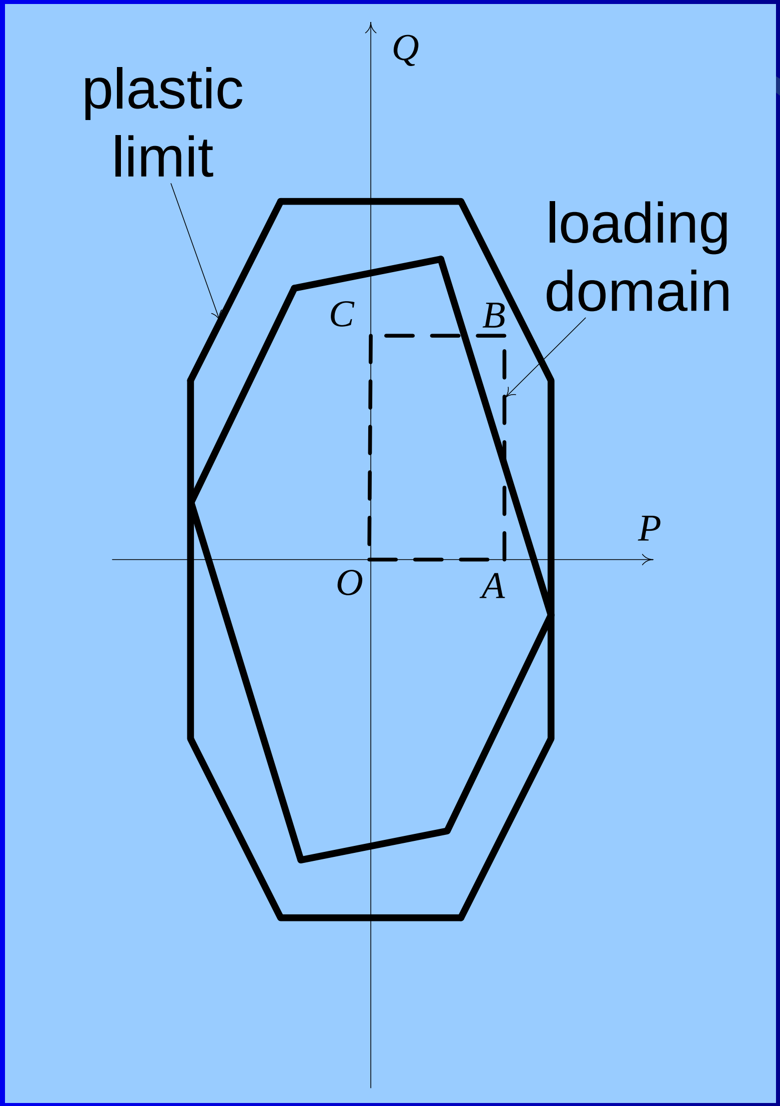
-Shakedown analysis for a given loading domain, example:

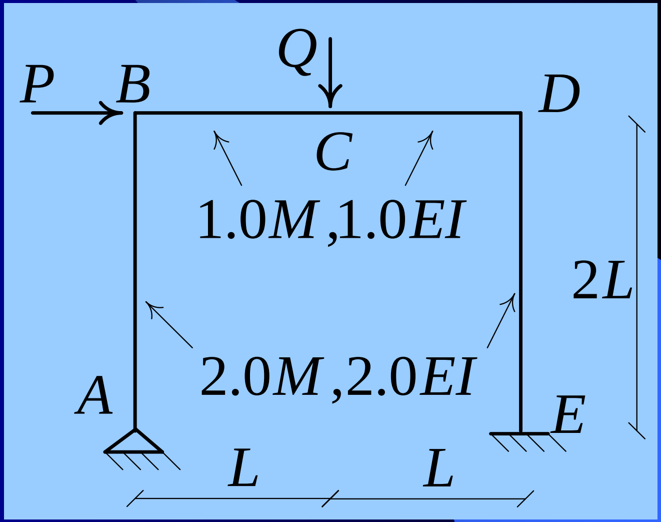
$$O:\begin{cases} P=0 \\ Q=0 \end{cases}$$

$$A:\begin{cases} P=\dfrac{1.5M_0}{L} \\ Q=0 \end{cases}$$

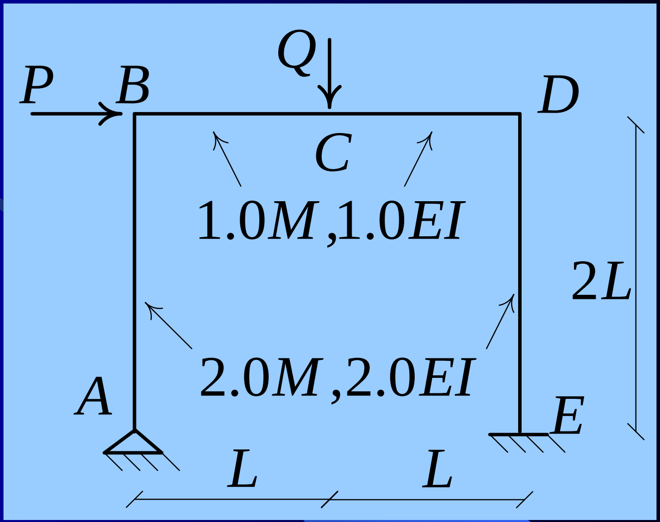
$$B:\begin{cases} P=\dfrac{1.5M_0}{L} \\ Q=\dfrac{2.5M_0}{L} \end{cases}$$

$$C:\begin{cases} P=0 \\ Q=\dfrac{2.5M_0}{L} \end{cases}$$





Shakedown analysis for a given loading domain, example, Continue…:



-Elastic solution:

$$M_{BA} = M_{BC} = 0.43038PL - 0.20886QL$$

$$M_{CB} = M_{CD} = -0.06329PL + 0.31013QL$$

$$M_{DC} = M_{DE} = -0.55696PL - 0.17089QL$$

$$M_{ED} = 1.01266PL + 0.03797QL$$

Shakedown analysis for a given loading domain, example, Continue…:

-Elastic solution against Loading:

$$O:\begin{cases} P=0 \\ Q=0 \end{cases}$$

$$M_{BA}=M_{BC}=0$$

$$M_{CB}=M_{CD}=0$$

$$M_{DC}=M_{DE}=0$$

$$M_{ED}=0$$

Shakedown analysis for a given loading domain, example, Continue…:

-Elastic solution against Loading:

$$A: \begin{cases} P = \dfrac{1.5M_0}{L} \\ Q = 0 \end{cases}$$

$$M_{BA} = M_{BC} = 0.64557 M_0$$

$$M_{CB} = M_{CD} = -0.09494 M_0$$

$$M_{DC} = M_{DE} = -0.83544 M_0$$

$$M_{ED} = 1.51899 M_0$$

Shakedown analysis for a given loading domain, example, Continue…:

-Elastic solution against Loading:

$$B:\begin{cases} P=\dfrac{1.5M_0}{L} \\ Q=\dfrac{2.5M_0}{L} \end{cases}$$

$$M_{BA}=M_{BC}=0.12342M_0$$

$$M_{CB}=M_{CD}=0.68039M_0$$

$$M_{DC}=M_{DE}=-1.26267M_0$$

$$M_{ED}=1.61392M_0$$

Shakedown analysis for a given loading domain, example, Continue…:

-Elastic solution against Loading:

$$C:\begin{cases} P = 0 \\ Q = \dfrac{2.5M_0}{L} \end{cases}$$

$$M_{BA} = M_{BC} = -0.52215M_0$$

$$M_{CB} = M_{CD} = 0.77533M_0$$

$$M_{DC} = M_{DE} = -0.42723M_0$$

$$M_{ED} = 0.09493M_0$$

Shakedown analysis for a given loading domain, example, Continue…:

-Maximum and minimum of Elastic solutions:

$$M_{eB}^{\max} = 0.64557 M_0 \qquad M_{eB}^{\min} = -0.52215 M_0$$

$$M_{eC}^{\max} = 0.77533 M_0 \qquad M_{eC}^{\min} = -0.09494 M_0$$

$$M_{eD}^{\max} = 0 \qquad M_{eD}^{\min} = -1.26267 M_0$$

$$M_{eE}^{\max} = 1.61392 M_0 \qquad M_{eE}^{\min} = 0$$

Shakedown analysis for a given loading domain, example, Continue…:

- Shakedown condition:

$$\mathbf{s}_r + \mathbf{s}_e^{max} < \mathbf{s}_0$$

$$\mathbf{s}_r + \mathbf{s}_e^{min} > -\mathbf{s}_0$$

$$M_{rB} + \mu_{inc} 0.64557 M_0 < M_0 \qquad M_{rB} - \mu_{inc} 0.52215 M_0 > -M_0$$

$$M_{rC} + \mu_{inc} 0.77533 M_0 < M_0 \qquad M_{rC} - \mu_{inc} 0.09494 M_0 > -M_0$$

$$M_{rD} + \mu_{inc} \times 0 < M_0 \qquad M_{rD} - \mu_{inc} 1.26267 M_0 > -M_0$$

$$M_{rE} + \mu_{inc} 1.61392 M_0 < 2M_0 \qquad M_{rE} + \mu_{inc} \times 0 > -2M_0$$

- Self equilibrium condition for residual forces using virtual work:

$$\begin{cases} -M_{rB} + 2M_{rC} - M_{rD} = 0 \\ M_{rB} - M_{rD} + M_{rE} = 0 \end{cases} \qquad \begin{cases} M_{rC} = 0.5 M_{rB} + 0.5 M_{rD} \\ M_{rE} = M_{rD} - M_{rB} \end{cases}$$

Shakedown analysis for a given loading domain, example, Continue…:

$$(I): M_{rB} + \mu_{inc} 0.64557 M_0 < M_0$$

$$(II): M_{rB} - \mu_{inc} 0.52215 M_0 > -M_0$$

$$(III): 0.5M_{rB} + 0.5M_{rD} + \mu_{inc} 0.77533 M_0 < M_0$$

$$(IV): 0.5M_{rB} + 0.5M_{rD} - \mu_{inc} 0.09494 M_0 > -M_0$$

$$(V): M_{rD} + \mu_{inc} \times 0 < M_0$$

$$(VI): M_{rD} - \mu_{inc} 1.26267 M_0 > -M_0$$

$$(VII): M_{rD} - M_{rB} + \mu_{inc} 1.61392 M_0 < 2M_0$$

$$(IIX): M_{rD} - M_{rB} + \mu_{inc} \times 0 > -2M_0$$

$$(I \,\&\, II): -1 + 0.52215\mu_{inc} < M_{rB}/M_0 < 1 - 0.64557\mu_{inc}$$

$$(III \,\&\, IV): -2 + 0.18988\mu_{inc} < M_{rB}/M_0 + M_{rD}/M_0 < 2 - 1.55066\mu_{inc}$$

$$(V \,\&\, VI): -1 + 1.26267\mu_{inc} < M_{rD}/M_0 < 1$$

$$(VII \,\&\, IIX): -2 < M_{rD}/M_0 - M_{rB}/M_0 < 2 - 1.61392\mu_{inc}$$

Shakedown analysis for a given loading domain, example, Continue…:

$$(I\ \&\ II):(IX):-1+0.52215\mu_{inc}<M_{rB}/M_0<1-0.64557\mu_{inc}$$

$$(III\ \&\ IV):(X):-2+0.18988\mu_{inc}<M_{rB}/M_0+M_{rD}/M_0<2-1.55066\mu_{inc}$$

$$(V\ \&VI):(XI):-1+1.26267\mu_{inc}<M_{rD}/M_0<1$$

$$(VII\ \&\ IIX):(XII):-2<M_{rD}/M_0-M_{rB}/M_0<2-1.61392\mu_{inc}$$

$$(IX\ \&\ XII):-3+0.52215\mu_{inc}<M_{rD}/M_0<3-2.25949\mu_{inc}$$

$$(X\ \&\ XII):-2+0.09494\mu_{inc}<M_{rD}/M_0<2-1.58229\mu_{inc}$$

$$(XI):-1+1.26267\mu_{inc}<M_{rD}/M_0<1$$

$$\max\begin{Bmatrix}-3+0.52215\mu_{inc}\\-2+0.09494\mu_{inc}\\-1+1.26267\mu_{inc}\end{Bmatrix}<\min\begin{Bmatrix}3-2.25949\mu_{inc}\\2-1.58229\mu_{inc}\\1\end{Bmatrix}$$

Shakedown analysis for a given loading domain, example, Continue…:

$$\max\begin{Bmatrix}-3+0.52215\mu_{inc}\\-2+0.09494\mu_{inc}\\-1+1.26267\mu_{inc}\end{Bmatrix}<\min\begin{Bmatrix}3-2.25949\mu_{inc}\\2-1.58229\mu_{inc}\\1\end{Bmatrix}$$

$$\mu_{inc}<\begin{Bmatrix}2.157\\2.376\\7.660\\2.124\\2.385\\31.599\\1.136\\1.054\\1.584\end{Bmatrix}$$

$$\mu_{inc}=1.054>1$$

The given loading domain is safe against incremental collapse.

Shakedown analysis for a given loading domain, example, Continue…:

- Preventing from plastic fatigue:

$$\mathbf{s}_e^{\max} - \mathbf{s}_e^{\min} < 2\mathbf{s}_0$$

$$\mathbf{M}_e^{\max} - \mathbf{M}_e^{\min} < 2\mathbf{M}_0/a$$

- Suppose: a=1.15

$$\begin{cases} M_{eB}^{\max} = 0.64557 M_0 \\ M_{eB}^{\min} = -0.52215 M_0 \end{cases}$$

$$\mu_{ftg}\left(0.64557 M_0 - \left(-0.52215 M_0\right)\right) < 2 M_0 / 1.15$$

$$\begin{cases} M_{eC}^{\max} = 0.77533 M_0 \\ M_{eC}^{\min} = -0.09494 M_0 \end{cases}$$

$$\mu_{ftg}\left(0.77533 M_0 - \left(-0.09494 M_0\right)\right) < 2 M_0 / 1.15$$

$$\begin{cases} M_{eD}^{\max} = 0 \\ M_{eD}^{\min} = -1.26267 M_0 \end{cases}$$

$$\mu_{ftg}\left(0 - \left(-1.26267 M_0\right)\right) < 2 M_0 / 1.15$$

$$\begin{cases} M_{eE}^{\max} = 1.61392 M_0 \\ M_{eD}^{\min} = 0 \end{cases}$$

$$\mu_{ftg}\left(1.61392 M_0 - 0\right) < 4 M_0 / 1.15$$

Shakedown analysis for a given loading domain, example, Continue…:

$$\mu_{ftg} < 1.489$$

$$\mu_{ftg} < 1.998$$

$$\mu_{ftg} < 1.377$$

$$\mu_{ftg} < 2.155$$

$$\mu_{ftg} = 1.377$$

$$\mu_{shk} = \min \begin{Bmatrix} \mu_{inc} = 1.054 \\ \mu_{ftg} = 1.377 \end{Bmatrix}$$

$$\mu_{shk} = 1.054$$

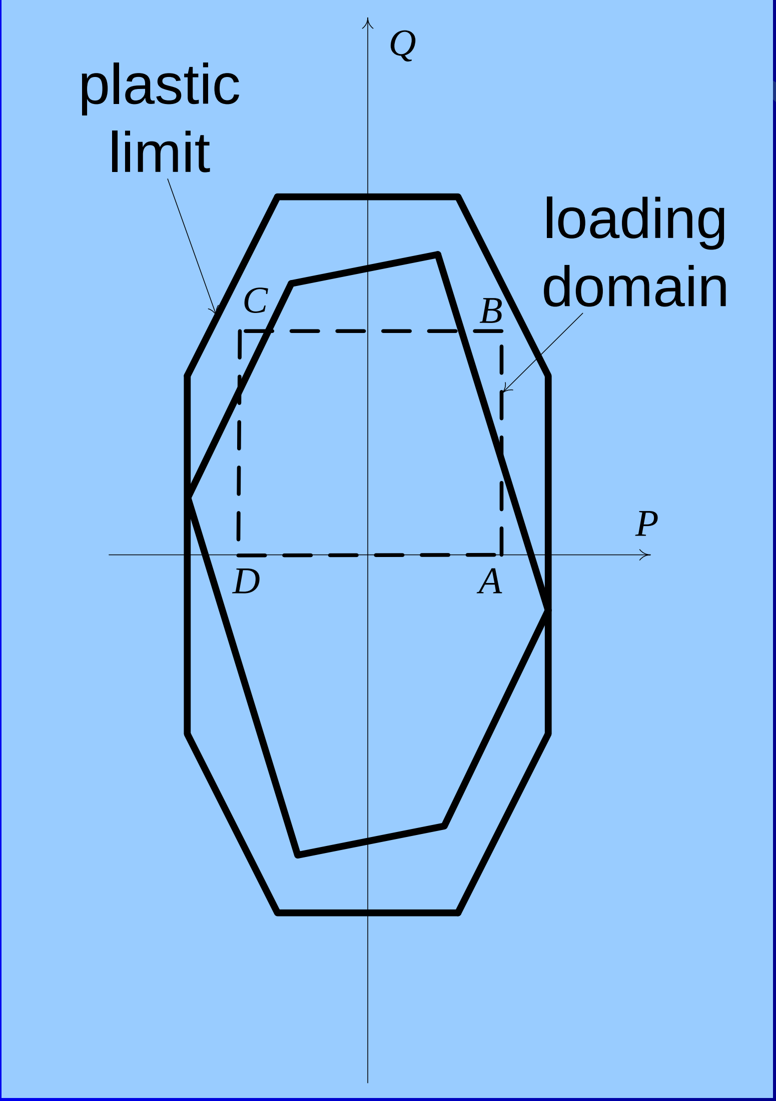
-Shakedown analysis for a given loading domain, example:

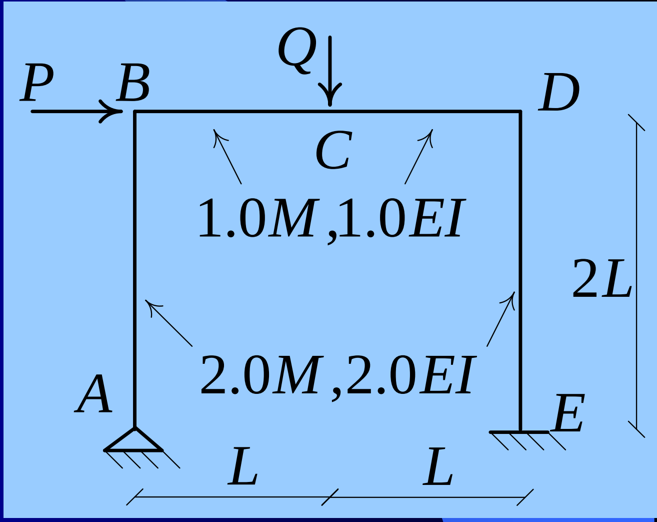
$$D:\begin{cases} P = -\dfrac{1.5M_0}{L} \\ Q = 0 \end{cases}$$

$$A:\begin{cases} P = \dfrac{1.5M_0}{L} \\ Q = 0 \end{cases}$$

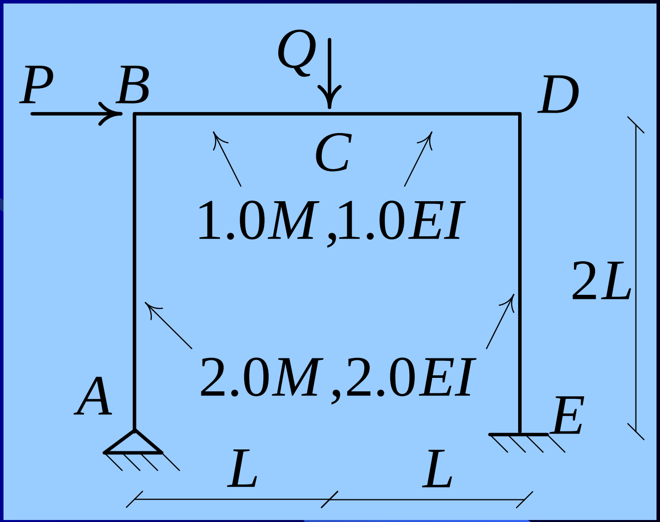
$$B:\begin{cases} P = \dfrac{1.5M_0}{L} \\ Q = \dfrac{2.5M_0}{L} \end{cases}$$

$$C:\begin{cases} P = -\dfrac{1.5M_0}{L} \\ Q = \dfrac{2.5M_0}{L} \end{cases}$$





Shakedown analysis for a given loading domain, example, Continue…:



-Elastic solution:

$$M_{BA} = M_{BC} = 0.43038PL - 0.20886QL$$

$$M_{CB} = M_{CD} = -0.06329PL + 0.31013QL$$

$$M_{DC} = M_{DE} = -0.55696PL - 0.17089QL$$

$$M_{ED} = 1.01266PL + 0.03797QL$$

Shakedown analysis for a given loading domain, example, Continue…:

-Elastic solution against Loading:

$$D:\begin{cases} P=-\dfrac{1.5M_0}{L} \\ Q=0 \end{cases}$$

$$M_{BA}=M_{BC}=-0.64557M_0$$

$$M_{CB}=M_{CD}=0.09494M_0$$

$$M_{DC}=M_{DE}=0.83544M_0$$

$$M_{ED}=-1.51899M_0$$

Shakedown analysis for a given loading domain, example, Continue…:

-Elastic solution against Loading:

$$A:\begin{cases} P = \dfrac{1.5M_0}{L} \\ Q = 0 \end{cases}$$

$$M_{BA} = M_{BC} = 0.64557M_0$$

$$M_{CB} = M_{CD} = -0.09494M_0$$

$$M_{DC} = M_{DE} = -0.83544M_0$$

$$M_{ED} = 1.51899M_0$$

Shakedown analysis for a given loading domain, example, Continue…:

-Elastic solution against Loading:

$$B:\begin{cases} P = \dfrac{1.5M_0}{L} \\ Q = \dfrac{2.5M_0}{L} \end{cases}$$

$$M_{BA} = M_{BC} = 0.12342M_0$$

$$M_{CB} = M_{CD} = 0.68039M_0$$

$$M_{DC} = M_{DE} = -1.26267M_0$$

$$M_{ED} = 1.61392M_0$$

Shakedown analysis for a given loading domain, example, Continue…:

-Elastic solution against Loading:

$$C:\begin{cases} P=-\dfrac{1.5M_0}{L} \\ Q=\dfrac{2.5M_0}{L} \end{cases}$$

$$M_{BA}=M_{BC}=-1.16772M_0$$

$$M_{CB}=M_{CD}=0.87026M_0$$

$$M_{DC}=M_{DE}=0.40822M_0$$

$$M_{ED}=-1.42407M_0$$

Shakedown analysis for a given loading domain, example, Continue…:

-Maximum and minimum of Elastic solutions:

$$M_{eB}^{\max} = 0.64557 M_0$$

$$M_{eB}^{\min} = -1.16772 M_0$$

$$M_{eC}^{\max} = 0.87026 M_0$$

$$M_{eC}^{\min} = -0.09494 M_0$$

$$M_{eD}^{\max} = 0.83544 M_0$$

$$M_{eD}^{\min} = -1.26267 M_0$$

$$M_{eE}^{\max} = 1.61392 M_0$$

$$M_{eE}^{\min} = -1.51899 M_0$$

Shakedown analysis for a given loading domain, example, Continue…:

- Shakedown condition:

$$s_r + s_e^{max} < s_0$$

$$s_r + s_e^{min} > -s_0$$

$$M_{rB} + \mu_{inc} 0.64557 M_0 < M_0$$

$$M_{rB} - \mu_{inc} 1.16772 M_0 > -M_0$$

$$M_{rC} + \mu_{inc} 0.87026 M_0 < M_0$$

$$M_{rC} - \mu_{inc} 0.09494 M_0 > -M_0$$

$$M_{rD} + \mu_{inc} \times 0.83544 M_0 < M_0$$

$$M_{rD} - \mu_{inc} 1.26267 M_0 > -M_0$$

$$M_{rE} + \mu_{inc} 1.61392 M_0 < 2M_0$$

$$M_{rE} - \mu_{inc} \times 1.51899 M_0 > -2M_0$$

- Self equilibrium condition for residual forces using virtual work:

$$\begin{cases} -M_{rB} + 2M_{rC} - M_{rD} = 0 \\ M_{rB} - M_{rD} + M_{rE} = 0 \end{cases}$$

$$\begin{cases} M_{rC} = 0.5M_{rB} + 0.5M_{rD} \\ M_{rE} = M_{rD} - M_{rB} \end{cases}$$

Shakedown analysis for a given loading domain, example, Continue…:

$$(I): M_{rB} + \mu_{inc} 0.64557 M_0 < M_0$$

$$(II): M_{rB} - \mu_{inc} 1.16772 M_0 > -M_0$$

$$(III): 0.5 M_{rB} + 0.5 M_{rD} + \mu_{inc} 0.87026 M_0 < M_0$$

$$(IV): 0.5 M_{rB} + 0.5 M_{rD} - \mu_{inc} 0.09494 M_0 > -M_0$$

$$(V): M_{rD} + \mu_{inc} \times 0.83544 M_0 < M_0$$

$$(VI): M_{rD} - \mu_{inc} 1.26267 M_0 > -M_0$$

$$(VII): M_{rD} - M_{rB} + \mu_{inc} 1.61392 M_0 < 2M_0$$

$$(IIX): M_{rD} - M_{rB} - \mu_{inc} \times 1.51899 M_0 > -2M_0$$

$$(I \,\&\, II): -1 + 1.16772 \mu_{inc} < M_{rB} / M_0 < 1 - 0.64557 \mu_{inc}$$

$$(III \,\&\, IV): -2 + 0.18988 \mu_{inc} < M_{rB} / M_0 + M_{rD} / M_0 < 2 - 1.74052 \mu_{inc}$$

$$(V \,\&\, VI): -1 + 1.26267 \mu_{inc} < M_{rD} / M_0 < 1 - 0.83544 \mu_{inc}$$

$$(VII \,\&\, IIX): -2 + 1.51899 \mu_{inc} < M_{rD} / M_0 - M_{rB} / M_0 < 2 - 1.61392 \mu_{inc}$$

Shakedown analysis for a given loading domain, example, Continue…:

$$(I \& II):(IX):-1+1.16772\mu_{inc} < M_{rB}/M_0 < 1-0.64557\mu_{inc}$$

$$(III \& IV):(X):-2+0.18988\mu_{inc} < M_{rB}/M_0 + M_{rD}/M_0 < 2-1.74052\mu_{inc}$$

$$(V \& VI):(XI):-1+1.26267\mu_{inc} < M_{rD}/M_0 < 1-0.83544\mu_{inc}$$

$$(VII \& IIX):(XII):-2+1.51899\mu_{inc} < M_{rD}/M_0 - M_{rB}/M_0 < 2-1.61392\mu_{inc}$$

$$(IX \& XII):-3+2.68671\mu_{inc} < M_{rD}/M_0 < 3-2.25949\mu_{inc}$$

$$(X \& XII):-2+0.85444\mu_{inc} < M_{rD}/M_0 < 2-1.67722\mu_{inc}$$

$$(XI):-1+1.26267\mu_{inc} < M_{rD}/M_0 < 1-0.83544\mu_{inc}$$

$$\max\begin{Bmatrix} -3+2.68671\mu_{inc} \\ -2+0.85444\mu_{inc} \\ -1+1.26267\mu_{inc} \end{Bmatrix} < \min\begin{Bmatrix} 3-2.25949\mu_{inc} \\ 2-1.67722\mu_{inc} \\ 1-0.83544\mu_{inc} \end{Bmatrix}$$

Shakedown analysis for a given loading domain, example, Continue…:

$$\max\begin{Bmatrix} -3+2.68671\mu_{inc} \\ -2+0.85444\mu_{inc} \\ -1+1.26267\mu_{inc} \end{Bmatrix} < \min\begin{Bmatrix} 3-2.25949\mu_{inc} \\ 2-1.67722\mu_{inc} \\ 1-0.83544\mu_{inc} \end{Bmatrix}$$

$$\mu_{inc} < \begin{Bmatrix} 1.213 \\ 1.975 \\ 1.136 \\ 1.606 \\ 1.580 \\ 1.775 \\ 1.136 \\ 1.020 \\ 0.953 \end{Bmatrix}$$

$$\mu_{inc} = 0.953 < 1$$

The given loading domain isn't safe against incremental collapse.

Shakedown analysis for a given loading domain, example, Continue…:

- Preventing from plastic fatigue:

$$\mathbf{s}_e^{\max} - \mathbf{s}_e^{\min} < 2\mathbf{s}_0$$

$$\mathbf{M}_e^{\max} - \mathbf{M}_e^{\min} < 2\mathbf{M}_0/a$$

- Suppose: a=1.15

$$\begin{cases} M_{eB}^{\max} = 0.64557M_0 \\ M_{eB}^{\min} = -1.16772M_0 \end{cases}$$

$$\mu_{ftg}\left(0.64557M_0 - \left(-1.16772M_0\right)\right) < 2M_0/1.15$$

$$\begin{cases} M_{eC}^{\max} = 0.87026M_0 \\ M_{eC}^{\min} = -0.09494M_0 \end{cases}$$

$$\mu_{ftg}\left(0.877026M_0 - \left(-0.09494M_0\right)\right) < 2M_0/1.15$$

$$\begin{cases} M_{eD}^{\max} = 0.83544M_0 \\ M_{eD}^{\min} = -1.26267M_0 \end{cases}$$

$$\mu_{ftg}\left(0.83544M_0 - \left(-1.26267M_0\right)\right) < 2M_0/1.15$$

$$\begin{cases} M_{eE}^{\max} = 1.61392M_0 \\ M_{eD}^{\min} = -1.51899M_0 \end{cases}$$

$$\mu_{ftg}\left(1.61392M_0 - \left(-1.51899M_0\right)\right) < 4M_0/1.15$$

Shakedown analysis for a given loading domain, example, Continue…:

$$\mu_{ftg} < 0.959$$

$$\mu_{ftg} < 1.789$$

$$\mu_{ftg} < 0.829$$

$$\mu_{ftg} < 1.110$$

$$\mu_{ftg} = 0.829$$

$$\mu_{shk} = \min\begin{Bmatrix} \mu_{inc} = 0.953 \\ \mu_{ftg} = 0.829 \end{Bmatrix}$$

$$\mu_{shk} = 0.829$$