# DEFLECTIONS OF BEAMS AND FRAMES

- INTRODUCTION : When a structure is loaded, its stressed elements deform. In a truss, bars in tension elongate and bars in compression shorten. Beams bend. As this deformation occur, the structure changes shape and points on the structure displace. Although these deflections are normally small, as a part of the total design the engineer must verify that these deflections are within the limits specified by the governing design code to ensure that the structure is serviceable. Large deflections cause cracking of non structural elements such as plaster ceiling, tile walls or brittle pipes. Since the magnitude of deflections is also a measure of a member's stiffness, limiting deflections also ensures that excessive vibrations of building floors.
- In this chapter we consider several methods of computing deflections and slopes at points along the axis of beams and frames. These methods are based on the differential equation of the elastic curve of a beam. This equation relates curvature at a point along beam's longitudinal axis to the bending moment at that point and the properties of the cross section and the material. If the elastic curve seems difficult to establish, it is suggested that the moment diagram for the beam or frame be drawn first. A positive moment tends to bend a beam concave upward. Likewise a negative moment bend the beam concave downward.

Therefore if the moment diagram is known, it will be easy to construct the elastic curve. In particular, there must be an inflection point at the point where the curve changes from concave down to concave up, since this is a zero moment.



## DOUBLE INTEGRATION METHOD

The double integration method is a procedure to establish the equations for slope and deflection at points along the elastic curve of a loaded beam. The equations are derived by integrating the differential equation of the elastic curve twice. The method assumes that all deformations are produced by moment.



The slope of the curve at point A $\frac{dy}{dx} = \tan \theta$ If the angles are small, the<br/>slope can be written $\frac{dy}{dx} = \theta$ From the geometry of<br/>triangular segment ABO $\rho.d\theta = ds$ 

Dividing each side by ds

 $\frac{1}{\rho} = \frac{d\theta}{ds} = \psi$ 

 $d\Theta/ds$  represents the change in slope per unit length of distance along the curve, is called curvature and denoted by symbol  $\Psi$ . Since slopes are small in actual beams ds=dx we can express the curvature,

$$\psi = \frac{d\theta}{dx} = \frac{1}{\rho}$$

Differentiating both sides of the second equation

 $\frac{d\theta}{dx} = \frac{d^2y}{dx^2}$ 

Relationship between bending moment and curvature for pure bending remains valid for general transverse loadings.

$$\psi = \frac{1}{\rho} = \frac{d\theta}{dx} = \frac{d^2 y}{dx^2} = \frac{M}{EI}$$

Example : For the cantilever beam in figure establish the equations for slope and deflection by the double integration method. Also determine the magnitude of the slope  $\Theta_B$  and deflection  $\Delta_B$  at the tip of the cantilever. EI is constant.



$$M = -P(L - x)$$
  

$$\frac{d^{2}y}{dx^{2}} = \frac{M}{EI} = \frac{-P(L - x)}{EI}$$
  

$$\frac{dy}{dx} = \frac{-PLx}{EI} + \frac{Px^{2}}{2EI} + C_{1}$$
  

$$y = \frac{-PLx^{2}}{2EI} + \frac{Px^{3}}{6EI} + C_{1}x + C_{2}$$

Boundary conditions

when 
$$\dots x = 0 \dots y = 0 \dots$$
 then  $\dots C_2 = 0$   
when  $\dots x = 0 \dots \frac{dy}{dx} = 0 \dots$  then  $\dots C_1 = 0$   
 $\theta = \frac{dy}{dx} = \frac{-PLx}{EI} + \frac{Px^2}{2EI}$   
 $y = \frac{-PLx^2}{2EI} + \frac{Px^3}{6EI}$ 

$$\theta_B = \frac{-PL^2}{2EI}$$
$$\Delta_B = \frac{-PL^3}{3EI}$$

Example : Use the double integration method, establish the equations for slope and deflection for the uniformly loaded beam in figure. Evaluate the deflection at mid-span and the slope at support A. EI is constant.



and the slope at A substitute x=0

$$\theta_A = \frac{-qL^3}{24 EI}$$

## MOMENT-AREA METHOD

There are two moment-area theorems. These theorems were developed by Otto Mohr and later stated formally by Charles E. Greene in 1872. These theorems provide a semi-graphical technique for determining the slope of the elastic curve and its deflection due to bending. They are particularly advantageous when used to solve problems involving beams especially those subjected to serious of concentrated loadings or having segments with different moments of inertia. The first theorem is used to calculate a change in slope between two points on the elastic curve. The second theorem is used to compute the vertical distance (called a tangential deviation) between a point on the elastic curve and a line tangent to the elastic curve at a second point. These quantities are illustrated in the following figure.



**Derivation of the moment-area theorems**: The relation between slope and moment and bending stiffness EI is

$$\frac{d\theta}{dx} = \frac{M}{EI}$$
$$d\theta = \frac{M}{EI}dx$$

It states that the change in slope of the tangents on either side of the element dx is equal to the area under the M/EI diagram. Integrating from point A on the elastic curve to point B we have,

$$\Delta \theta_{AB} = \int_{A}^{B} \frac{M}{EI} dx$$

The ordinates of the moment curve must be divided by the bending stiffness EI to produce M/EI curve. The last equation forms the basis for the first moment-area theorem.

The change in slope between any two points on a continuous elastic curve is equal to the area under the M/EI curve between these points.







The tangential deviation dt can be expressed by

$$dt = d\theta . x$$

Substitute  $d\Theta$  in the equation,

 $dt = (\frac{M}{EI}dx).x$ 

To evaluate  $t_{BA}$  we must sum all increments of dt by integrating the contribution of all the infinitesimal segment between points A and B.

$$t_{BA} = \int_{A}^{B} dt = \int_{A}^{B} \frac{M.x}{EI} dx$$

Remembering that the quantity Mdx/EI represents an infinitesimal area under the M/EI diagram and that x is the distance from the area to point B, we can interpret the integral as the moment about point B of the area under the M/EI diagram between points A and B

#### This result constitutes the second moment-area theorem.

The tangential deviation at a point B on a continuous elastic curve from a tangent drawn to the elastic curve at a second point A is equal to the moment about B of the area under the M/EI diagram between the two points.

Example : Compute the slope and deflection at the tip of the cantilever beam. EI is constant.



Tangent at A is always horizontal.

$$\theta_{B} = \theta_{A} + \Delta \theta_{AB} = \Delta \theta_{AB}$$

$$\Delta \theta_{AB} = \frac{1}{2} L \left( \frac{-PL}{EI} \right) = \frac{-PL^2}{2EI} \Longrightarrow \theta_{AB} = \frac{-PL^2}{2EI}$$

The slope at mid-point :

$$\Delta \theta_{AC} = \frac{1}{2} \left( \frac{-PL}{EI} + \frac{-PL}{2EI} \right) \frac{L}{2} = \frac{-3PL^2}{8EI}$$
$$\theta_{AC} = \frac{-3PL^2}{8EI}$$

$$t_{BA} = v_B = \frac{1}{2}L\frac{-PL}{EI}\frac{2L}{3} = \frac{-PL^3}{3EI}$$

Minus sign indicates that the tangent lies above elastic curve

Example : Determine the deflection at points B and C of the beam shown below. Values for the moment of inertia of each segment are  $I_{AB}$  = 8.10<sup>6</sup> mm<sup>4</sup>, and  $I_{BC}$  = 4.10<sup>6</sup> mm<sup>4</sup>. Take E = 200 GPa,



By inspection the moment diagram for the beam is a rectangle. If we construct the M/EI diagram relative to 
$$I_{BC}$$
. The couple causes the beam to deflect concave up. We are required to find the vertical displacements at B and C. These displacements can be related directly to the deviations between the tangents.

$$\Delta_B = t_{BA} = \frac{250}{EI_{BC}} * 4 * 2 = \frac{2000}{EI_{BC}} = \frac{2000}{200.10^9 * 4.10^{-6}} = 0.0025m$$

$$\Delta_C = t_{CA} = \frac{250}{EI_{BC}} * 4 * 5 + \frac{500}{EI_{BC}} * 3 * 1.5 =$$

$$\frac{7250}{EI_{BC}} = 0.00906m = 9.06mm$$

Since both answers are positive, they indicate that points B and C lie above the tangent at A.

Example : Determine the slope at point C of the beam. EI is constant.



$$\begin{aligned} \theta_{C} &= \theta_{A} - \Delta \theta_{AC} \\ \theta_{C} &= \frac{t_{BA}}{L} - \Delta \theta_{AC} \\ \theta_{C} &= \frac{t_{BA}}{L} - \Delta \theta_{AC} \\ \Delta \theta_{AC} &= \frac{1}{2} \left( \frac{4}{EI} * 2 \right) = \frac{4}{EI} \\ t_{BA} &= \frac{1}{2} \left( \frac{12}{EI} * 6 \right) * 4 + \frac{1}{2} \left( \frac{12}{EI} * 2 \right) * 2 * \frac{2}{3} = \frac{160}{EI} \\ \theta_{C} &= \frac{160}{8EI} - \frac{4}{EI} = \frac{16}{EI} \dots \theta_{A} = \frac{20}{EI} \end{aligned}$$

Location of maximum deflection occurs at the point where the slope of the beam is zero. So, the change in slope must be  $\Theta_A$ , between the support and the point of the max. deflection. let the base of the triangle be x

$$\theta_A = \frac{20}{EI} = \frac{x^2}{EI} \Longrightarrow x = 2\sqrt{5} = 4.47m$$

$$\frac{\Delta'}{6} = \frac{t_{BA}}{8} \to \Delta' = \frac{3}{4} t_{BA}$$
$$\Delta_{\text{max}} = \Delta' - t_{DA} = \frac{3}{4} \frac{160}{EI} - \frac{40\sqrt{5}}{3EI} = \frac{90.186}{EI}$$

Example : Determine the deflection at point C of the beam. E= 200 GPa, I=250 10<sup>-6</sup> m<sup>4</sup>



### CONJUGATE BEAM METHOD

The method was first presented by Otto Mohr in 1860. This method relies only the principles of statics and hence its application will be more familiar. The basis for the method comes from the similarity between both

$$\frac{dQ}{dx} = -q \qquad \qquad \frac{dM}{dx} = -Q \qquad or......\frac{d^2M}{dx^2} = q$$

which relate a beam's internal shear and moment to its applied loading, and

$$\frac{d\theta}{dx} = \frac{M}{EI} \qquad \qquad \frac{d^2 y}{dx^2} = \frac{M}{EI}$$

which relate the slope and deflection of its elastic curve to the internal moment divided by EI. Or integrating

$Q = \int (-q) dx$	$M = -\int \int (-q) dx$
$\theta = \int \left(\frac{M}{EI}\right) dx$	$y = \int \int \left(\frac{M}{EI}\right) dx$

These equations indicate that the shear in a beam can be obtained by integrating the load once and the moment by integrating the load twice. Since the curvature of a beam is proportional to the bending moment, slope and deflection of the beam can be obtained by successively integrating the moment.

We want to replace the integration indicated in the equations by drawing the shear and bending moment diagrams. To do this we will consider a beam having the same length as the real beam, bot referred to here as the **conjugate beam**. It is loaded with the M/EI diagram of the real beam. Shear and moment diagrams of the conjugate beam represent one and two integrations, respectively of the M/EI diagram of the real beam. We thus conclude that shear and moment diagrams of the conjugate beam represent the slope and deflection of the real beam. Now we can therefore state two theorems related to conjugate beam, namely

**Theorem 1**: The slope at a point in the real beam is equal to the shear at the corresponding point in the conjugate beam.

**Theorem 2**: The displacement at a point in the real beam is equal to the moment at the corresponding point in the conjugate beam.

Since each of the previous equations requires integration it is important that the proper boundary conditions be used when they are applied.





If we treat positive values of the M/EI diagram applied to the beam as a distributed load acting upward and negative values of M/EI as a downward load, positive shear denotes a negative slope and a negative shear a positive slope. Further negative values of moment indicate a downward deflection and positive values of moment an upward deflection.

As a rule, neglecting axial force, statically determinate real beams have statically indeterminate conjugate beams; and statically indeterminate real beams, as in the last figure above become unstable conjugate beams. Although this occurs, the M/EI loading will provide the necessary "equilibrium" to hold the conjugate beam stable.

Example : Determine the slope and deflection of point B of the steel beam shown in figure. The reactions are given in the figure. Take E=200 GPa, I=333 10<sup>6</sup> mm<sup>4</sup>



101.2 kNm



M/EI diagram is negative, so the distributed load acts downward

$$-\frac{232.76}{EI} + Q_B = 0$$

$$Q_B = \frac{232.76}{EI} = \frac{232.76}{200.10^6 * 333.10^{-6}}$$

$$Q_B = 0.00349$$

$$\theta_B = -0.00349 rad$$

$$M_{B} + \frac{232.76}{EI} * 7.667 = 0$$

$$M_{B} = \Delta_{B} = \frac{-1784.5}{EI} = \frac{-1784.5}{200.10^{6} * 333.10^{-6}} = \Delta_{B} = -0.0268 m$$

Positive shear indicates negative slope, and the negative moment of the beam indicates downward displacement.

Example : Determine the maximum deflection of the steel beam shown below. Reactions are given. I= 60.10<sup>6</sup> mm<sup>4</sup>



M

$$\frac{45}{EI} * 6.71 - \frac{1}{2} \left( \frac{2 * 6.71}{EI} * 6.71 \right) * \frac{6.71}{3} + M = 0$$
  
$$\Delta_{\text{max}} = M = \frac{-201 \cdot 2}{EI} = \frac{-201 \cdot 2}{200 \cdot 10^{-6} * 60 \cdot 10^{-6}} = -0.0168 \quad m$$

Example: Determine the displacement of the pin at B and the slope of the each beam segment connected to the pin for the beam shown in the figure. E=200 GPa I=135.10<sup>6</sup> mm<sup>4</sup>



M/EI diagram has been drawn in parts using the principle of superposition. Notice that negative regions of this diagram develop a downward distributed load and the positive regions have a distributed load that acts upward.

To determine  $\Theta_{B}^{r}$  the conjugate beam is sectioned just to the right of B and shear force is computed



The internal moment at B yields the displacement of the pin

$$\sum M = 0$$

$$M_{B} + \frac{276}{EI} * 3.076 - \frac{593.74}{EI} * 4.60 = 0$$

$$\Delta_{B} = M_{B} = \frac{1884.8}{EI} = \frac{1884.8}{200.10^{6} * 135.10^{-6}} = 0.070 m$$



The slope  $\Theta_{B}^{I}$  can be found from a section of beam just to the left of B thus





## EXTERNAL WORK AND STRAIN ENERGY

Most energy methods are based on the conservation of the energy which states that work done by all the external forces acting on a structure, W, is transformed into internal work or strain energy U, which is developed when the structure deforms. If the material's elastic limit is not exceeded, the elastic strain energy will return the structure to its undeformed state when the loads are removed.

W=U External work done by a force : When a force F undergoes a dW = F dxdisplacement dx in the same direction as the force, the work done is,  $W = \int F dx$ If the total displacement is x, the work becomes  $W = \int_{0}^{x} F dx = \frac{P}{\Delta} \int_{0}^{\Delta} x dx = \frac{P\Delta}{2}$ F Represents the triangular area X

Suppose that P is already applied to the bar and that another force F' is now applied, so the bar deflects further by an amount  $\Delta'$ , the work done by P (not F') when the bar undergoes the further deflection  $\Delta'$  is then

$$W' = P.\Delta'$$

P does not change its magnitude. Work is simply the force magnitude (P) times the displacement ( $\Delta$ ')



**External work done by a moment:** The work of a moment is defined by the product of the magnitude of the moment and the angle  $d\Theta$  through which it rotates.

If the total angle of rotation is  $\Theta$  radians, the work becomes

As in the case of force, if the moment is applied to a structure gradually, from zero to M, the work is then

$$W = \frac{1}{2} M \theta$$

However if the moment is already applied to the structure and other loadings further distort the structure by an amount  $\Theta'$ , then the M rotates  $\Theta'$ , and the work is

$$W' = M.\theta'$$

 $dW = M.d\theta$ 

 $W = \int_{0}^{\theta} M.d\theta$ 

Strain Energy due to an axial force



$$\sigma = E \cdot \varepsilon$$

$$\frac{N}{A} = E \frac{\Delta}{L}$$

$$\Delta = \frac{NL}{EA}$$

$$U = \frac{1}{2} N \cdot \Delta = \frac{N^2 L}{2 EA}$$



#### Principle of work and energy

Application of this method is limited to only a few select problems. It will be noted that only one load may be applied to the structure and only the displacement under the load can be obtained.

Example: Using principle of work and energy, determine the deflection at the tip of the cantilever beam.



If we take a deformable structure, and apply a series of loads to it, it will cause internal loads and internal displacements



Principle of work and energy states

$$\sum P_i \Delta_i = \sum N_j \Delta L_j$$
$$\sum P_i \Delta_i = \int M d\theta$$

Work of external forces

Work of internal forces

#### Virtual work method (Unit Load Method)

Virtual work is a procedure for computing a single component of deflection at any point on a structure. The method permits the designer to include in deflection computations the influence of support settlements, temperature change, and fabrication errors. To compute a component of deflection by the method of virtual work, the designer applies a unit force to the structure at the point and in the direction of the desired displacement. This force is often called a *dummy load*, the displacement it will undergo is produced by other effects. These other effects include the real loads, temperature change, support settlement and so forth. In other words, the deflected shape due to the real forces is assumed to be the virtual deflected shape of the dummy structure, and the principle of work and energy is applied.

#### Application of unit force method to trusses



 $\Delta L = \frac{N.L}{EA}$  $\Delta L = \alpha . \Delta T . L$  $\Delta L$ 

Due to real loads

Due to temperature change

Given fabrication error



not to the right as assumed

**Example:** The truss shown below is distorted during fabrication because member BC is 12.7 mm short. What vertical deflection is introduced at point E because of this misfit.





$$1.\Delta_{E}^{V} = \sum N_{j}.\Delta L_{j} = -0.75 * (-12.7 mm) = 9.525 mm$$

Example: Determine the vertical displacement that occurs at point B as a result temperature change of  $+30^{\circ}$  C in members AD and DC. The coefficient of thermal expansion is  $11.7*10^{-6}$   $1/^{\circ}$ C.



 $\Delta L_{AD} = \Delta L_{DC} = 11.7 * 10^{-6} * 30 * 6.25$  $= 2.194 * 10^{-3} m = 2.194 mm$ 



MEMBER	dL	Nv	Nv.dL
AD	2,194	-0,625	-1,371
DC	2,194	-0,625	-1,371
DB	0	1	0
AB	0	0,375	0
BC	0	0,375	0
			-2,743

$$1.\Delta_{\mathbf{B}}^{V} = \sum N_{j}.\Delta L_{j} = -2.74 \, mm$$

Example: Determine the change in member length for member BD of the truss so that, when combined with the prescribed loading, will produce a net vertical displacement of zero at point D. E= 200 GPa.





Member	L ( m )	Ν	Nv	EA	N.Nv.L/EA	
AB	5	10	-1,25	200000	-0,0003124	
AD	4,125	-8,25	1,031	160000	-0,0002193	
BC	5,657	-16,97	-1,414	200000	0,0006787	
BD	2	6	1,75	100000	0,00021	
DC	4,472	-8,944	1,118	160000	-0,0002795	
				sum=	7,744E-05	

$$1.\Delta_D^V = \sum \frac{N.Nv.L}{EA} = 0.00007744m$$

0.07744*mm* 



 $1.\Delta = \sum N_{j}.\Delta L_{j}$ 1\*0.07744 = (-1.75)\* $\Delta L$  $\Delta L = -0.0443 mm$ 

We want to shorten the member so that upward

deflection at point D be 0.07744 mm.

Application of unit force method to beams and frames



$$\Delta = \int_{0}^{L} \frac{M_{q} \cdot M_{p} dx}{EI}$$

Integral of the product of moment functions is needed. Prepare an integral table to simplify the calculations

$$\Delta = \int_{0}^{l} f(x) F(x) \frac{dx}{EI}$$



$$f(x): i.... const$$
.  
 $F(x): k... const$ .  $\rightarrow \Delta = \frac{L.i.k}{EI}$ 

$$f(x):\frac{i}{L}x....linear$$

$$F(x):\frac{k}{L}x....linear \rightarrow \Delta = \frac{L.i.k}{3 EI}$$







$$f(x):\frac{i}{L}x$$
$$F(x):\frac{k}{L^{2}}x^{2}.... \rightarrow \Delta = \frac{L.i.k}{4 EI}$$

**Example:** Determine the rotation at A and B to an applied moment M on the beam as shown.



Example: What is the vertical deflection of the free end of the cantilever beam.





Example: Determine both the vertical and the horizontal deflection at A for the frame shown. E=200 GPa I=200 10<sup>6</sup> mm<sup>4</sup>

$$\Delta_{A}^{v} = \int_{0}^{L} \frac{M_{p} \cdot M_{q}}{EI} dx = \frac{L}{6EI} k \cdot (i_{1} + 2i_{2}) + \frac{L \cdot i \cdot k}{EI} = \frac{1}{EI} \left(\frac{2}{6}100 * (2 + 8) + 5 * 100 * 4\right) = \frac{2333 \cdot .33}{EI} = 0.058 m$$

T

$$\Delta_{A}^{h} = \int_{0}^{L} \frac{M_{p} M_{q}}{EI} dx = \frac{L}{2EI} i k = \frac{1}{EI} \frac{5}{2} \cdot 100 \cdot 5 = \frac{1250}{EI} = 0.031 m$$

#### Deflection of structures consisting of flexural members and axially loaded members

Example: Find the horizontal displacement at C. E=200 GPa I=150.10<sup>6</sup> mm<sup>4</sup> A=50 mm<sup>2</sup>



The internal work for the given structure consists of two parts : the work due to bending of the frame and the work due to axial deformation of the cable.

$$\Delta_{c}^{h} = \int \frac{M_{p} \cdot M_{q}}{EI} dx + \frac{N_{p} \cdot N_{q} \cdot L}{EA} = \frac{1}{EI} \left(\frac{3}{3} * 120 * 3 + \frac{6}{3} * 480 * 3 - \frac{6}{4} * 360 * 3\right) + \frac{80 * 0.5 * 4}{EA}$$
$$= \frac{1620}{30000} + \frac{160}{10000} = 0.070 \, m = 70 \, mm$$



What are the relative rotation and the vertical displacement at  $\ensuremath{\mathcal{C}}$ 



What are the rotation and vertical displacement at D



### MAXWELL'S RECIPROCAL THEOREM

Let us consider the beam in the figure. Because of the load  $F_1$  the beam deflects an amount  $\delta_{11}F_1$  at point 1 and an amount  $\delta_{21}F_1$  at point 2.



Where  $\delta_{11}$  and  $\delta_{21}$  are the deflections at points 1 and 2 due to a unit load at point 1

 $\delta_{i\,i}\,$  deflection at point i due to a unit load at j

- i place of deflection
- j place of unit load

Now we will formulate an expression for the work due to  $\mathsf{F}_1$  and  $\mathsf{F}_2$ 

Apply the forces  $\mathsf{F}_1$  and  $\mathsf{F}_2$  simultaneously the resulting work can be written

$$W = \frac{1}{2} \left( F_1 \Delta_1 + F_2 \Delta_2 \right)$$

 $W = \frac{1}{2} \left( F_1 \left( \delta_{11} F_1 + \delta_{12} F_2 \right) + F_2 \left( \delta_{21} F_1 + \delta_{22} F_2 \right) \right)$  $W = \frac{1}{2} \left( \delta_{11} F_1^2 + \left( \delta_{12} + \delta_{21} \right) F_1 F_2 + \delta_{22} F_2^2 \right)$ 

If we apply  $F_1$  first the amount of work performed is

$$W^{1} = \frac{1}{2}F_{1} \cdot \delta_{11}F_{1} = \frac{1}{2}\delta_{11}F_{1}^{2}$$

Next we apply  $\mathsf{F}_2$  to the beam on which  $\mathsf{F}_1$  is already acting. The additional work resulting from the application of  $\mathsf{F}_2$ 

$$W^{2} = F_{1}(\delta_{12}F_{2}) + \frac{1}{2}F_{2}(\delta_{22}F_{2})$$

 $\frac{1}{2}$  factor is absent on the first term because  $F_1$  remains constant at its full value during the entire displacement. The total work due to  $F_1$  and  $F_2$ .

In a linear system, the work performed by two forces is independent of the order in which the forces are applied. Hence the two works must be equal.

$$\frac{1}{2} \left( \delta_{11} F_1^2 + (\delta_{12} + \delta_{21}) F_1 F_2 + \delta_{22} F_2^2 \right) = \frac{1}{2} \left( \delta_{11} F_1^2 + \delta_{22} F_2^2 \right) + \delta_{12} F_1 F_2$$
  
$$\frac{1}{2} \left( \delta_{12} + \delta_{21} \right) = \delta_{12}$$
  
$$\left( \delta_{12} + \delta_{21} \right) = 2 \delta_{12}$$
  
$$\delta_{12} = \delta_{21}$$

This relationship is known as Maxwell's reciprocal theorem





Consider two different loading on a linear elastic structure. The virtual work done by the forces of the first system acting through the displacements of the second system is equal to the virtual work done by the forces of the second system acting through the corresponding displacements of the first system.

#### CASTIGLIANO'S FIRST THEOREM



Since the strain energy U stored in a deformed structure is equal to the work performed by the external loads

 $\frac{\partial U}{\partial P_i} = \Delta_i$ 

The partial derivative of the strain energy in a structure with respect to one of the external loads acting on the structure is equal to the displacement at that force in the direction of the force. This relation is known as Castigliano's theorem



$$\frac{\partial U}{\partial R_1} = 0 \qquad \qquad \frac{\partial U}{\partial R_2} = 0$$

This theorem can be used to analyze the redundant structures



This method is called as the method of least work. Redundant must have a value that will make the strain energy in the structure a minimum. The known values of the displacements at certain points can be used for additional equations to the equations of equilibrium.

The strain energy for an axially loaded member

The strain energy for a truss

$$U = \frac{N^2 L}{2 AE} \qquad \qquad U = \sum \frac{N^2 L}{2 AE}$$

To find a joint displacement of a truss Castigliano's theorem can be used

$$\delta_{i} = \frac{\partial U}{\partial P_{i}} = \sum N \left( \frac{\partial N}{\partial P_{i}} \right) \frac{L}{AE}$$

 $\delta_{I}$ : joint displacement in the direction of the external force  $P_{i}$ 

N : axial force in member

E : Modulus of elasticity A: Cross sectional area

To take the partial derivative, force  $\mathsf{P}_{\mathsf{i}}$  must be replaced by a variable force

Example: Determine the vertical displacement of joint C of the truss shown in the figure. The cross sectional area of each member is  $A = 400 \text{ mm}^2$  and E = 200 GPa



A vertical force P is applied to do truss at joint C, since this is where the vertical displacement is to be determined.

Since P does not actually exist as a real load on the truss. We require p=0 in the table.

Member	Ν	$\partial N/\partial P$	N (P=0)	L (m)	$N.\partial N/\partial P.L$
AB	0.667P+2	0.677	2	8	10.67
AC	-0.833P+2.5	-0.833	2.5	5	-10.41
BC	-0.833P-2.5	-0.833	-2.5	5	10.41
				-	

10.67 kN.m



0.000133m = 0.133mm



Example: Determine the vertical displacement of joint C of the truss shown in the figure. Assume that A=325mm<sup>2</sup>, E=200 GPa .



Member	Ν	$\partial N/\partial P$	N (P=20)	L	N.∂N/∂
AB	13.333+0.333P	0.333	20	3	19.98
BC	6.667+0.667P	0.667	20	3	40.02
CD	6.667+0.667P	0.667	20	3	40.02
DE	-9.429-0.943P	-0.943	-28.289	4.24	113.11
EF	-13.333-0.333P	-0.333	-20	3	19.98
FA	-18.856-0.471P	-0.471	-28.276	4.24	56.47
BF	13.333+0.333P	0.333	20	3	19.98
BE	9.429-0.471P	-0.471	0	4.24	0
CE	Р	1	20	3	60

N.∂N/∂P.L

Sum 369.56

$$\delta_{c}^{v} = \sum N \frac{\partial N}{\partial P} \frac{L}{AE} = \frac{369.56}{AE} =$$

$$\frac{369.56}{325*10^{-6}*200*10^{6}} =$$

0.005685m = 5.685mm

find the horizontal displacement of point D



Membe			Ν		N.∂N/∂P
r	Ν	$\partial N / \partial P$	(P=20)	L	L
AB	20+P	1	20	3	60
BC	20+P	1	20	3	60
CD	20+P	1	20	3	60
DE	-28.28	0	-28.28	4.24	0
EF	-20	0	-20	3	0
FA	-28.28	0	-28.28	4.24	0
BF	20	0	20	3	0
BE	0	0	0	4.24	0
CE	20	0	20	3	0

 $\delta_{c}^{h} = \sum N \frac{\partial N}{\partial P} \frac{L}{AE} = \frac{180}{AE} =$ 

$$\frac{180}{325*10^{-6}*200*10^6} =$$

0.002769m = 2.77mm

#### CASTIGLIANO'S THEOREM FOR BEAMS AND FRAMES

The strain energy for beam or frame member is given by

 $U_{i} = \int \frac{M^2}{2EI} dx$ 

Substituting this equation into the Castigliano's first theorem

$$\Delta_{i} = \frac{\partial U_{i}}{\partial P_{i}} = \frac{\partial}{\partial P} \int \frac{M^{2}}{2EI} dx = \int_{0}^{l} M\left(\frac{\partial M}{\partial P_{i}}\right) \frac{dx}{EI}$$

 $\Delta$  = External displacement

P= External force applied to the beam or frame in the direction of the desired displacement

M= Internal moment in the beam or frame, expressed as function of x and caused by both the force P and the real loads.

E= Modulus of elasticity

I= Moment of inertia of cross sectional area computed about the neutral axis.

If the slope  $\Theta$  at a point is to be determined, the partial derivative of the internal moment M with respect to an external couple moment M' acting at the point must be computed.

$$\theta = \int_{0}^{l} M\left(\frac{\partial M}{\partial M'}\right) \frac{dx}{EI}$$

The above equations are similar to those used for the method of virtual work except the partial derivatives replace moments due to unit loads like the case of trusses, slightly more calculations is generally required to determine the partial derivatives and apply Castigliano's theorem rather than use the method of virtual force method. Example : determine the displacement of point B of the beam shown in figure. E=200 GPa I=500\*10<sup>6</sup> mm<sup>4</sup>



The moment at an arbitrary point on the beam and its derivative

$$M = -6x^{2} + (P+120)x - (10P+600)$$
$$\frac{\partial M}{\partial P} = x - 10$$

Setting P=0 yields

 $M = -6x^{2} + 120x - 600$  $\frac{\partial M}{\partial P} = x - 10$ 

A vertical force P is placed on the beam at B



Castigliano's theorem

$$\delta_B = \int M \frac{\partial M}{\partial P} \frac{dx}{EI} = \int_0^{10} \left(-6x^2 + 120x - 600\right) (x - 10) \frac{dx}{EI}$$

$$=\frac{1}{EI}\left|-\frac{6}{4}x^{4}+60x^{3}-900x^{2}+6000x\right|_{0}^{10}=\frac{15000}{EI}$$

 $=\frac{15000}{500*10^{-6}*200*10^{6}}=0.15m=150mm$ 



Example : determine the vertical displacement of point C of the beam. Take E=200 GPa I=150\*10<sup>6</sup> mm<sup>2</sup>





$$< x < 8m....M = 64 + 4P - (8 + 0.5P)x...\frac{\partial P}{\partial P} = 4 - 0.5x...M = 144 - 18x$$

4

$$\delta_{c}^{ver} = \int_{0}^{4} (34x - 4x^{2}) \frac{x}{2} \frac{dx}{EI} + \int_{4}^{8} (144 - 18x)(4 - 0.5x) \frac{dx}{EI}$$
$$= \frac{1}{2EI} \left| \frac{34}{3} x^{3} - x^{4} \right|_{0}^{4} + \frac{1}{EI} \left| 576x - 72x^{2} + 3x^{3} \right|_{4}^{8} = \frac{426.67}{EI}$$
$$= \frac{426.67}{200 \times 10^{6} \times 150 \times 10^{-6}} = 0.0142m = 14.2mm$$





$$\frac{640}{EI} + \frac{1}{EI} \left[ 160 * 5 - \frac{27}{2} * 25 + \frac{1.1}{3} * 125 \right] + \frac{1}{EI} \left[ 210 * (10 - 5) - \frac{42}{2} (100 - 25) + \frac{2.1}{3} (100 * -125) \right] = \frac{1235 .83}{EI}$$

Example : determine the reactions for the propped cantilever beam.





Reaction at point C is taken as redundant  $M_A=PL/2-RL$ 







BENDING MOMENT DIAGRAM (KNM)